AUTOMORPHISM GROUP OF A BOTT-SAMELSON-DEMAZURE-HANSEN VARIETY

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Abstract. Let G be a simple, adjoint, algebraic group over the field of complex numbers, B be a Borel subgroup of G containing a maximal torus T of G, w be an element of the Weyl group W and $X(w)$ be the Schubert variety in G/B corresponding to w. Let $Z(w, \underline{i})$ be the Bott-Samelson-Demazure-Hansen variety corresponding to a reduced expression $\underline{i}$ of w.

In this article, we compute the connected component $Aut^0(Z(w, \underline{i}))$ of the automorphism group of $Z(w, \underline{i})$ containing the identity automorphism. We show that $Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to B if and only if $w^{-1}(\alpha_0) < 0$, where α_0 is the highest root. If w_0 denotes the longest element of W, then we prove that $Aut^0(Z(w_0, \underline{i}))$ is a parabolic subgroup of G. It is also shown that this parabolic subgroup depends very much on the chosen reduced expression $\underline{i}$ of w_0 and we describe all parabolic subgroups of G that occur as $Aut^0(Z(w_0, \underline{i}))$. If G is simply laced, then we show that for every $w \in W$, and for every reduced expression $\underline{i}$ of w, $Aut^0(Z(w, \underline{i}))$ is a quotient of the parabolic subgroup $Aut^0(Z(w_0, \underline{j}))$ of G for a suitable choice of a reduced expression $\underline{j}$ of w_0 (see Theorem 7.3).

Keywords: Automorphism group, Bott-Samelson-Demazure-Hansen variety, Tangent Bundle.

1. Introduction

Let G be a simple algebraic group over the field $\mathbb{C}$ of complex numbers of adjoint type. We fix a maximal torus T of G and let $W = N_G(T)/T$ denote the Weyl group of G with respect to T. We denote the set of roots of G with respect to T by R. Let B^+ be a Borel subgroup of G containing T. Let B be the Borel subgroup of G opposite to B^+ determined by T. That is, $B = n_0 B^+ n_0^{-1}$, where n_0 is a representative in $N_G(T)$ of the longest element w_0 of W. Let $R^+ \subset R$ be the set of positive roots of G with respect to the Borel subgroup B^+. Note that the set of roots of B is equal to the set $R^- := -R^+$ of negative roots. We use the notation $\beta > 0$ for $\beta \in R^+$ and $\beta < 0$ for $\beta \in R^-$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of all simple roots in R^+, where n is the rank of G. The simple reflection in the Weyl group corresponding to a simple root α is denoted by s_α. For simplicity of notation, the simple reflection corresponding to a simple root α_i is denoted by s_i. For any simple root α, we denote the fundamental weight corresponding to α by ω_α. Let α_0 denote the highest root and ρ denote the half sum of all positive roots, which is also same as the sum of all fundamental weights.

For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. Given a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, \ldots, i_r)$, we denote by $Z(w, \underline{i})$ the desingularization of the Schubert variety $X(w)$, which is now known as the Bott-Samelson-Demazure-Hansen variety. It was first introduced by

Bott and Samelson in a differential geometric and topological context (see [3]). Demazure in [6] and Hansen in [9] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote any Bott-Samelson-Demazure-Hansen variety by a BSDH-variety.

The construction of the BSDH-variety $Z(w, \underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w. So, it is natural to ask that for a given $w \in W$ whether the BSDH-varieties corresponding to two different reduced expressions of w are isomorphic? This article deals with the study of the automorphism group of the BSDH-varieties in order to answer this question.

We recall the following notation before describing the main results:

Let $\mathfrak{g}$ denote the Lie algebra of G, $\mathfrak{h}$ be the Lie algebra of T, $\mathfrak{b}$ be the Lie algebra of B. Let $X(T)$ denote the group of all characters of T.

We have $X(T)\otimes\mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$. The positive definite W-invariant bilinear form on $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of $\mathfrak{g}$ is denoted by $(\ , \ )$. We use the notation $\langle \ , \ \rangle$ to denote $\langle \nu, \alpha \rangle = \frac{2(\nu,\alpha)}{(\alpha,\alpha)}$ for $\nu \in X(T) \otimes \mathbb{R}$ and $\alpha \in R$.

Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, let $\underline{i} := (i_1, \ldots, i_r)$. Set

$$J^{'}(w, \underline{i}) := \{l \in \{1, 2, \ldots, r\} : \langle \alpha_{i_l}, \alpha_{i_k} \rangle = 0 \text{ for all } k < l\}$$

$$J(w, \underline{i}) := \{\alpha_{i_l} : l \in J^{'}(w, \underline{i})\} \subset S.$$

Note that the simple reflections $\{s_{i_j} : j \in J'(w, \underline{i})\}$ commute with each other. Let $W_{J(w,\underline{i})}$ be the subgroup of W generated by $\{s_j \in W \mid \alpha_j \in J(w, \underline{i})\}$. Let

$$P_{J(w,\underline{i})} := BW_{J(w,\underline{i})}B$$

be the corresponding standard parabolic subgroup of G. By abuse of notation, here $W_{J(w,\underline{i})}$ in the definition of the parabolic subgroup $P_{J(w,\underline{i})}$ means any lift of elements of $W_{J(w,\underline{i})}$ to $N_G(T)$. Let $N = |R^+|$. Further, let $Aut^0(Z(w, \underline{i}))$ be the connected component of the automorphism group of $Z(w, \underline{i})$ containing the identity automorphism.

The main results of this article are (see Theorem 7.3):

(1) For any reduced expression $\underline{i}$ of w_0, $Aut^0(Z(w_0, \underline{i})) \simeq P_{J(w_0,\underline{i})}$.
(2) For any reduced expression $\underline{i}$ of w, $Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, $P_{J(w,\underline{i})} = P_{J(w_0,\underline{j})}$ for any reduced expression $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \ldots, j_N)$ and $(j_1, j_2, \ldots, j_r) = \underline{i}$.
(3) If G is simply laced, $Aut^0(Z(w, \underline{i}))$ is a quotient of $Aut^0(Z(w_0, \underline{j}))$, where $\underline{j}$ is as in (2).
(4) If G is simply laced, $Aut^0(Z(w, \underline{i})) \simeq P_{J(w,\underline{i})}$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, we have $P_{J(w,\underline{i})} = P_{J(w_0,\underline{j})}$ where $\underline{j}$ is as in (2).
(5) The rank of $Aut^0(Z(w, \underline{i}))$ is at most the rank of G.

Consider the left action of T on G/B. Note that $X(w)$ is T-stable. Since T is a reductive group, studying the semi-stable points of $X(w)$ for T-linearized line bundles is an interesting problem related to Geometric Invariant Theory. By [15, Lemma 2.1], the condition

$w^{-1}(\alpha_0) < 0$ is equivalent to the Schubert variety $X(w^{-1})$ having semi-stable points for the choice of the T-linearized line bundle $\mathcal{L}_{\alpha_0}$ associated to α_0. Corollary 7.5 is a formulation of the main results using semi-stable points.

The paper is organized as follows:

In Section 2, we recall the definition of the BSDH-varieties and some results on the cohomology of line bundles on Schubert varieties. The main results used here are the results of Demazure ([6] and [7]). A structure theorem for indecomposable B_α-modules is recalled from [1], where B_α is the intersection of B and the Levi subgroup of the minimal parabolic subgroup of G containing B corresponding to $\alpha \in S$. The important results recalled are from [16], which states that all i^{th} cohomology groups of $\mathcal{L}_\beta$ vanish on $X(w)$ for all $i \geq 2$ and for all $w \in W$ and for any positive root β. In the simply laced case in fact these cohomology groups vanish for all $i > 0$.

Section 3 begins with a detailed description of the BSDH-varieties as iterated $\mathbb{P}^1$-bundles. Using the results of [16], we conclude that higher cohomology groups (that is $i > 1$ in general and $i > 0$ in the simply laced case) of the tangent bundle of the BSDH-variety vanish (see Proposition 3.1). This implies that the BSDH-varieties are rigid for simply laced groups and their deformations are unobstructed in general.

Next three sections are more technical sections.

Section 4 is devoted to detailed computations involving the structure of H^0 and H^1 of the relative tangent bundle on $Z(w, \underline{i})$, where $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression for w and $\underline{i} = (i_1, \ldots, i_r)$. We analyze the zero weight spaces and the weight spaces corresponding to positive roots of the global sections of the relative tangent bundle and we prove that these spaces are multiplicity free. We also prove that $\mathfrak{b} \cap sl_{2,\alpha_{i_r}}$ is a $B_{\alpha_{i_r}}$-submodule of the global sections of the relative tangent bundle if and only if $X(s_{i_r}) \nsubseteq X(s_{i_1} \cdots s_{i_{r-1}})$. While proving this, we observe that its zero weight space is at most one-dimensional (see Lemma 4.3). Further, we prove that $sl_{2,\alpha_{i_r}}$ is a $B_{\alpha_{i_r}}$-submodule if and only if $\langle \alpha_{i_r}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq r-1$ (see Corollary 4.5). We conclude the section with a result on the H^1 of the relative tangent bundle.

In section 5, we discuss the action of the minimal parabolic subgroup $P_{\alpha_{i_1}}$ on the BSDH-variety $Z(w, \underline{i})$. We show that the homomorphism $f_{w_0} : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ of Lie algebras induced by the action of $P_{\alpha_{i_1}}$ is injective (see Lemma 5.1). We also prove that $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$ and any Borel (respectively, maximal toral) subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ is isomorphic to a Borel (respectively, maximal toral) subalgebra of $\mathfrak{g}$ (see Corollary 5.2).

In Section 6, we study the B-module of the global sections of the tangent bundle on the BSDH-variety $Z(w, \underline{i})$. We prove that the image $f_w(\mathfrak{h})$ is a maximal toral subalgebra of $H^0(Z(w, \underline{i}), T_{(w, \underline{i})})$ (see Proposition 6.1). Further, we show that $sl_{\alpha_{i_j}}$ is a $B_{\alpha_{i_j}}$-submodule of $H^0(Z(w, \underline{i}), T_{(w, \underline{i})})$ if and only if $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$ (see Proposition 6.3). We conclude Section 6 by proving that $H^0(Z(w, \underline{i}), T_{(w, \underline{i})})$ contains a Lie subalgebra $\mathfrak{b}'$ isomorphic to $\mathfrak{b}$ if and only if $w^{-1}(\alpha_0) < 0$ (see Proposition 6.4).

In Section 7, we prove the main results on the connected component $Aut^0(Z(w, \underline{i}))$ of the automorphism group of the BSDH-variety $Z(w, \underline{i})$ using the fact that the global sections

$H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ of the tangent bundle on $Z(w,\underline{i})$ is the Lie algebra of $Aut^0(Z(w,\underline{i}))$. More precisely, we prove that the Lie algebra $\mathfrak{p}_{J(w_0,\underline{i})}$ of $P_{J(w_0,\underline{i})}$ is isomorphic to $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$. We also prove that for any reduced expression $\underline{j} = (j_1, j_2, \ldots, j_N)$ of w_0 such that $(j_1, j_2, \ldots, j_r) = \underline{i}$ the homomorphism $H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ of Lie algebras induced by the fibration $Z(w_0,\underline{j}) \longrightarrow Z(w,\underline{i})$ is injective if and only if $w^{-1}(\alpha_0) < 0$. Further, we prove that if G is simply laced, the homomorphism $H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ as above is surjective (see Theorem 7.1). We also compute the kernel of this homomorphism (see Corollary 7.2). Using Theorem 7.1, we prove the main results of this article. Using Corollary 7.2, we describe the kernel of the natural homomorphism $Aut^0(Z(w_0,\underline{j})) \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups (see Corollary 7.4). Thus, we have a complete description of $Aut^0(Z(w,\underline{i}))$ for any reduced expression $\underline{i}$ of w in the simply laced case.

2. Preliminaries

Let $\{x_\beta : \beta \in R\} \cup \{h_\alpha : \alpha \in S\}$ be the Chevalley basis for $\mathfrak{g}$ corresponding to the root system R. For a simple root α, we denote by $\mathfrak{g}_\alpha$(respectively, $\mathfrak{g}_{-\alpha}$) the one-dimensional root subspace of $\mathfrak{g}$ spanned by x_α (respectively, $x_{-\alpha}$). Let $sl_{2,\alpha}$ denote the 3-dimensional Lie subalgebra of $\mathfrak{g}$ generated by x_α and $x_{-\alpha}$.

Let $\leq$ denote the partial order on $X(T)$ given by $\mu \leq \lambda$ if $\lambda - \mu$ is a non-negative integral linear combination of simple roots. We say that $\mu < \lambda$ if in addition $\lambda - \mu$ is non zero. We set $R^+(w) := \{\beta \in R^+ : w(\beta) \in R^-\}$. We refer to [11] and [12] for preliminaries on Lie algebras and algebraic groups.

For a simple root $\alpha \in S$, we denote by P_α the minimal parabolic subgroup of G generated by B and n_α, a lift of s_α in $N_G(T)$.

We recall that the BSDH-variety corresponding to a reduced expression $\underline{i}$ of $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is defined by

$$Z(w,\underline{i}) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}}{B \times \cdots \times B},$$

where the action of $B \times \cdots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ is given by $(p_1, \ldots, p_r)(b_1, \ldots, b_r) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{r-1}^{-1} \cdot p_r \cdot b_r)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$ and $\underline{i} = (i_1, i_2, \ldots, i_r)$ (see [6, p.73, Definition 1], [4, p.64, Definition 2.2.1] and [9]).

We note that for each reduced expression $\underline{i}$ of w, $Z(w,\underline{i})$ is a smooth projective variety. We denote both the natural birational surjective morphism from $Z(w,\underline{i})$ to $X(w)$ and the composition map $Z(w,\underline{i}) \longrightarrow X(w) \hookrightarrow G/B$ by ϕ_w.

Let $f_r : Z(w,\underline{i}) \longrightarrow Z(ws_{i_r}, \underline{i}')$ denote the map induced by the projection

$$P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_{r-1}}},$$

where $i' = (i_1, i_2, \ldots, i_{r-1})$. We note that f_r is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$-fibration.

Now, we recall some preliminaries on the BSDH-varieties and some application of Leray spectral sequences to compute the cohomology of line bundles on Schubert varieties. Good references for this are [4] and [14].

Let L_α denote the Levi subgroup of P_α containing T for $\alpha \in S$. We denote by B_α the intersection of L_α and B. Then L_α is the product of T and a homomorphic image G_α of $SL(2,\mathbb{C})$ via a homomorphism $\psi : SL(2,\mathbb{C}) \longrightarrow L_\alpha$ (see [14, II, 1.3]).

Let $B'_\alpha := B_\alpha \cap G_\alpha \subset L_\alpha$. We note that the morphism $G_\alpha/B'_\alpha \longrightarrow L_\alpha/B_\alpha$ induced by the inclusion is an isomorphism. Since $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ is an isomorphism, to compute the cohomology groups $H^i(P_\alpha/B, V)$ for any B-module V, we treat V as a B_α-module and we compute $H^i(L_\alpha/B_\alpha, V)$.

For a B-module V, let $\mathcal{L}(w,V)$ denote the restriction of the associated homogeneous vector bundle on G/B to $X(w)$. By abuse of notation we denote the pull back of $\mathcal{L}(w,V)$ via ϕ_w to $Z(w,\underline{i})$ also by $\mathcal{L}(w,V)$, when there is no cause for confusion. Then, for $j \geq 0$, we have the following isomorphism of B-linearized sheaves (see [14, II, p.366]):

$$R^j f_{r*}\mathcal{L}(w,V) = \mathcal{L}(ws_{i_r}, H^j(P_{\alpha_{i_r}}/B, \mathcal{L}(s_{\alpha_{i_r}}, V))). \qquad (Iso)$$

We use the following *ascending 1-step construction* as a basic tool in computing cohomology modules.

For $w \in W$, let $l(w)$ denote the length of w. Let γ be a simple root such that $l(w) = l(s_\gamma w) + 1$. Let $Z(w,\underline{i})$ be a BSDH-variety corresponding to a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, where $\alpha_{i_1} = \gamma$. Then, we have an induced morphism

$$g : Z(w,\underline{i}) \longrightarrow P_\gamma/B \simeq \mathbb{P}^1,$$

with fibres $Z(s_\gamma w, \underline{i}')$, where $\underline{i}' = (i_2, i_3, \ldots, i_r)$.

By an application of the Leray spectral sequence together with the fact that the base is $\mathbb{P}^1$, we obtain for every B-module V, the following exact sequence of P_γ-modules:

$$0 \to H^1(P_\gamma/B, R^{j-1}g_*\mathcal{L}(w,V)) \to H^j(Z(w,\underline{i}), \mathcal{L}(w,V)) \to H^0(P_\gamma/B, R^j g_*\mathcal{L}(w,V)) \to 0.$$

Since for any B-module V, the vector bundle $\mathcal{L}(w,V)$ on $Z(w,\underline{i})$ is the pull back of the homogeneous vector bundle from $X(w)$, we conclude that the cohomology modules

$$H^j(Z(w,\underline{i}),\ \mathcal{L}(w,V)) \cong H^j(X(w),\ \mathcal{L}(w,V))$$

(see [4, Theorem 3.3.4 (b)]), and are independent of the choice of the reduced expression $\underline{i}$. Hence we denote $H^j(Z(w,\underline{i}),\ \mathcal{L}(w,V))$ by $H^j(w,V)$. For a character λ of B, we denote the one-dimensional B-module corresponding to λ by $\mathbb{C}_\lambda$. Further, we denote the cohomology modules $H^j(Z(w,\underline{i}),\ \mathcal{L}(w,\mathbb{C}_\lambda))$ by $H^j(w,\lambda)$.

Rewriting the above short exact sequence using these simple notation, we have the following short exact sequence:

$$0 \to H^1(s_\gamma, H^{j-1}(s_\gamma w, V)) \to H^j(w,V) \to H^0(s_\gamma, H^j(s_\gamma w, V)) \to 0.$$

In this paper, the B-modules V we deal with satisfy $R^j g_* \mathcal{L}(w,V) = 0$ for all $j \geq 2$. Moreover, we use only the following two special cases of the above short exact sequence, which we denote by *SES*.

(1) $H^0(w,V) \simeq H^0(s_\gamma, H^0(s_\gamma w, V))$ for $j=0$.
(2) $0 \to H^1(s_\gamma, H^0(s_\gamma w, V)) \to H^1(w,V) \to H^0(s_\gamma, H^1(s_\gamma w, V)) \to 0$ for $j=1$.

Now, we recall the following result due to Demazure ([7], Page 1) on a short exact sequence of B-modules:

Lemma 2.1. *Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let ev denote the evaluation map $H^0(s_\alpha, \lambda) \longrightarrow \mathbb{C}_\lambda$. Then we have*

(1) *If $\langle \lambda, \alpha \rangle = 0$, then $H^0(s_\alpha, \lambda) \simeq \mathbb{C}_\lambda$.*
(2) *If $\langle \lambda, \alpha \rangle \geq 1$, then $\mathbb{C}_{s_\alpha(\lambda)} \hookrightarrow H^0(s_\alpha, \lambda)$ and there is a short exact sequence of B-modules:*
$$0 \longrightarrow H^0(s_\alpha, \lambda - \alpha) \longrightarrow H^0(s_\alpha, \lambda)/\mathbb{C}_{s_\alpha(\lambda)} \xrightarrow{ev} \mathbb{C}_\lambda \longrightarrow 0.$$
Further more, $H^0(s_\alpha, \lambda - \alpha) = 0$ when $\langle \lambda, \alpha \rangle = 1$.
(3) *Let $n = \langle \lambda, \alpha \rangle$. As a B-module, $H^0(s_\alpha, \lambda)$ has a composition series*
$$0 \subsetneq V_n \subsetneq V_{n-1} \subsetneq \ldots \subsetneq V_0 = H^0(s_\alpha, \lambda)$$
such that $V_i/V_{i+1} \simeq \mathbb{C}_{\lambda - i\alpha}$ for $i = 0, 1, \cdots, n-1$ and $V_n = \mathbb{C}_{s_\alpha(\lambda)}$.

We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for any $w \in W$ and $\lambda \in X(T) \otimes \mathbb{R}$. Note that $s_\alpha \cdot 0 = -\alpha$ for $\alpha \in S$. As a consequence of the exact sequences of Lemma 2.1, we can prove the following.

Let $w \in W$, α be a simple root, and set $v = ws_\alpha$.

Lemma 2.2. *If $l(w) = l(v) + 1$, then, we have*

(1) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(v, H^0(s_\alpha, \lambda))$ for all $j \geq 0$.*
(2) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*
(3) *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*
(4) *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.*

Proof. Choose a reduced expression of $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ with $\alpha_{i_r} = \alpha$. Hence $v = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ is a reduced expression for v. Let $\underline{i} = (i_1, i_2, \ldots, i_r)$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Now consider the morphism $f_r : Z(w, \underline{i}) \longrightarrow Z(v, \underline{i}')$ defined as above.

Proof of (1): Since $\langle \lambda, \alpha \rangle \geq 0$, we have $H^j(s_\alpha, \lambda) = 0$ for every $j > 0$. Hence using the isomorphism (Iso), we have $R^j f_{r*} \mathcal{L}(w, \lambda) = 0$ for every $j > 0$. Therefore, by [10, p.252, III, Ex 8.1] we have $H^i(w, \lambda) = H^i(v, H^0(s_\alpha, \lambda))$ for every $i \geq 0$.

Proof of (3): Since $\langle \lambda, \alpha \rangle \leq -2$, by using (Borel-Weil-Bott theorem) [7, Theorem 2 (c)] for $L_\alpha/B_\alpha (\simeq P_\alpha/B)$; we have $H^i(s_\alpha, \lambda) = 0$ for $i \neq 1$ and $H^1(s_\alpha, \lambda) = H^0(s_\alpha, s_\alpha \cdot \lambda)$. By (Iso), we have $R^j f_{r*} \mathcal{L}(w, \lambda) = 0$ for every $j \neq 1$. Hence by using Leray spectral sequence, we see that $H^{j+1}(w, \lambda) = H^j(v, R^1 f_{r*} \mathcal{L}(w, \lambda)) = H^j(v, H^1(s_\alpha, \lambda))$ (see [20, p.152, Section 5.8.6]). Hence $H^{j+1}(w, \lambda) = H^j(v, H^0(s_\alpha, s_\alpha \cdot \lambda))$ for every $j \geq 0$. Since $\langle s_\alpha \cdot \lambda, \alpha \rangle \geq 0$, by (1) we have

$H^j(v, H^0(s_\alpha, s_\alpha\cdot\lambda)) = H^j(w, s_\alpha\cdot\lambda)$ for every $j \geq 0$. Hence we have $H^{j+1}(w,\lambda) = H^j(w, s_\alpha\cdot\lambda)$ for every $j \geq 0$.

Proof of (2): It follows from (3) by interchanging the role of λ and $s_\alpha\cdot\lambda$, because $\langle s_\alpha\cdot\lambda, \alpha\rangle = -\langle\lambda,\alpha\rangle - 2$.

Proof of (4): If $\langle\lambda,\alpha\rangle = -1$, then $H^i(s_\alpha,\lambda) = 0$ for every $i \geq 0$ (see [14, p.218, Proposition 5.2(b)]). Now the proof of (4) follows by using similar arguments as in (1) and (3). $\square$

The following consequence of Lemma 2.2 will be used to compute cohomology modules in this paper.

Let $\pi : \widetilde{G} \longrightarrow G$ be the simply connected covering of G. Let $\widetilde{L}_\alpha$ (respectively, $\widetilde{B}_\alpha$ be the inverse image of L_α (respectively, B_α) in $\widetilde{G}$ under π.

Lemma 2.3. *Let V be an irreducible $\widetilde{L}_\alpha$-module. Let λ be a character of $\widetilde{B}_\alpha$. Then, we have*

(1) *As $\widetilde{L}_\alpha$-modules, $H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V\otimes\mathbb{C}_\lambda) \simeq V\otimes H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, \mathbb{C}_\lambda)$ for every $j \geq 0$.*
(2) *If $\langle\lambda,\alpha\rangle \geq 0$, $H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V\otimes\mathbb{C}_\lambda) = 0$ for every $j \geq 1$.*
(3) *If $\langle\lambda,\alpha\rangle \leq -2$, $H^0(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V\otimes\mathbb{C}_\lambda) = 0$, and*
$$H^1(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V\otimes\mathbb{C}_\lambda) \simeq V\otimes H^0(\widetilde{L}_\alpha/\widetilde{B}_\alpha, \mathbb{C}_{s_\alpha\cdot\lambda}).$$
(4) *If $\langle\lambda,\alpha\rangle = -1$, then $H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V\otimes\mathbb{C}_\lambda) = 0$ for every $j \geq 0$.*

Proof. Proof (1). By [14, p.53, I, Proposition 4.8] and [14, p.77, I, Proposition 5.12], for all $j \geq 0$, we have the following isomorphism of $\widetilde{L}_\alpha$-modules:
$$H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V\otimes\mathbb{C}_\lambda) \simeq V\otimes H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, \mathbb{C}_\lambda).$$

Proof of (2), (3) and (4) follows from Lemma 2.2 by taking $w = s_\alpha$ and the fact that $\widetilde{L}_\alpha/\widetilde{B}_\alpha \simeq P_\alpha/B$. $\square$

Recall the structure of indecomposable B_α-modules (see [1, p.130, Corollary 9.1]).

Lemma 2.4.

(1) *Any finite dimensional indecomposable $\widetilde{B}_\alpha$-module V is isomorphic to $V'\otimes\mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.*
(2) *Any finite dimensional indecomposable B_α-module V is isomorphic to $V'\otimes\mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.*

Proof. Proof of (1) follows from [1, p.130, Corollary 9.1].

Proof of (2) follows from the fact that every B_α-module can be viewed as a $\widetilde{B}_\alpha$-module via the natural homomorphism. $\square$

Now, we prove the following:

Corollary 2.5. *Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression for w such that $\langle\alpha_{i_j},\alpha_{i_r}\rangle = 0$ for every $j = 1, 2, \ldots, r-1$. Then, $H^0(w,\alpha_{i_r})$ is isomorphic to $H^0(s_{i_r},\alpha_{i_r})(\simeq sl_{2,\alpha_{i_r}})$.*

Proof. Since $L_{\alpha_{i_r}}/B_{\alpha_{i_r}} \hookrightarrow P_{\alpha_{i_r}}/B$ is an isomorphism, we have

$$sl_{2,\alpha_{i_r}} \simeq H^0(L_{\alpha_{i_r}}/B_{\alpha_{i_r}}, \alpha_{i_r}) \simeq H^0(s_{i_r}, \alpha_{i_r}).$$

We note that $sl_{2,\alpha_{i_r}}$ gets a natural B-module structure via the above isomorphism $sl_{2,\alpha_{i_r}} \simeq H^0(s_{i_r}, \alpha_{i_r})$.

Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$. If $l(v) = 0$, then $w = s_{i_r}$ and we are done. Otherwise, let $v^{'} = s_{i_2}\cdots s_{i_{r-1}}$. By induction on $l(v)$, we have

$$H^0(s_{i_2}\cdots s_{i_r}, \alpha_{i_r}) = H^0(s_{i_r}, \alpha_{i_r}).$$

By *SES*, we have

$$H^0(w, \alpha_{i_r}) = H^0(s_{i_1}, H^0(s_{i_2}\ldots s_{i_r}, \alpha_{i_r})) = H^0(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r})).$$

Since $\langle \alpha_{i_r}, \alpha_{i_1}\rangle = 0$ and $\langle -\alpha_{i_r}, \alpha_{i_1}\rangle = 0$, by Lemma 2.4, $H^0(s_{i_r}, \alpha_{i_r})$ is the trivial $B_{\alpha_{i_1}}$-module of dimension 3. Hence, the vector bundle $\mathcal{L}(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r}))$ on $X(s_{i_1}) \simeq \mathbb{P}^1$ is the trivial bundle of rank 3. Thus, we have

$$H^0(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r})) = H^0(s_{i_r}, \alpha_{i_r}).$$

□

We recall the following vanishing results from [16] (see [16, Corollary 3.6] and [16, Corollary 4.10]).

Lemma 2.6. *Let $w \in W$, and $\alpha \in R^+$. Then, we have*

(1) *$H^j(w, \alpha) = 0$ for all $j \geq 2$.*
(2) *If G is simply laced, $H^j(w, \alpha) = 0$ for all $j \geq 1$.*

Let $T_{G/B}$ denote the tangent bundle of the flag variety G/B. By abuse of notation, we denote the restriction $T_{G/B}$ to $X(w)$ by $T_{G/B}$. As we discussed in the introduction about the condition $w^{-1}(\alpha_0) < 0$, we state the following theorem from [16] (see [16, Theorem 3.7, Theorem 3.8 and Theorem 4.11]).

Theorem 2.7. *Let $w \in W$. Then*

(1) *$H^i(X(w), T_{G/B}) = 0$ for every $i \geq 1$.*
(2) *The adjoint representation $\mathfrak{g}$ of G is a B-submodule of $H^0(X(w), T_{G/B})$ if and only if $w^{-1}(\alpha_0) < 0$.*
(3) *If G is simply laced, $H^0(X(w), T_{G/B})$ is the adjoint representation $\mathfrak{g}$ of G if and only if $w^{-1}(\alpha_0) < 0$.*
(4) *Assume that G is simply laced and $X(w)$ is a smooth Schubert variety. Let $Aut^0(X(w))$ be the connected component of the automorphism group of $X(w)$ containing the identity automorphism. Let P_w denote the stabilizer of $X(w)$ in G. Let $\phi_w : P_w \longrightarrow Aut^0(X(w))$ be the homomorphism induced by the action of P_w on $X(w)$. Then, we have*
(i) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ is surjective.
(ii) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ is an isomorphism if and only if $w^{-1}(\alpha_0) < 0$.

3. Vanishing of the Higher Cohomology of the Tangent Bundle of $Z(w,\underline{i})$:

In this section, we prove that a BSDH variety has unobstructed deformations and it has no deformations whenever the group G is simply laced.

We recall that the BSDH-variety corresponding to a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is denoted by $Z(w,\underline{i})$ and we denote the tangent bundle of $Z(w,\underline{i})$ by $T_{(w,\underline{i})}$, where $\underline{i} = (i_1, i_2, \ldots, i_r)$.

Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Note that $l(v) = l(w)-1$. Consider the fibration $f_r : Z(w,\underline{i}) \longrightarrow Z(v,\underline{i}')$ as in Section 2. One can easily see that this fibration is the fibre product of $\pi_r : G/B \to G/P_{\alpha_{i_r}}$ and $\pi_r \circ \phi_v : Z(v,\underline{i}') \to G/P_{\alpha_{i_r}}$; namely, we have the following commutative diagram :

$$\begin{tikzcd}
Z(v,\underline{i}') \times_{G/P_{\alpha_{i_r}}} G/B = Z(w,\underline{i}) \arrow[r, "\phi_w"] \arrow[d, "f_r"] & G/B \arrow[d, "\pi_r"] \\
Z(v,\underline{i}') \arrow[r, "\pi_r\circ\phi_v"] & G/P_{\alpha_{i_r}}
\end{tikzcd}$$

The relative tangent bundle of π_r is the line bundle $\mathcal{L}(w_0, \alpha_{i_r})$. Hence the relative tangent bundle of f_r is $\phi_w^*\mathcal{L}(w_0, \alpha_{i_r})$. By taking the differentials of this smooth fibration f_r we obtain the following exact sequence:

$$0 \to \phi_w^*\mathcal{L}(w_0, \alpha_{i_r}) \to T_{(w,\underline{i})} \to f_r^*T_{(v,\underline{i}')} \to 0. \qquad (rel)$$

We use the above short exact sequence (rel) and Lemma 2.6 to prove the following:

Proposition 3.1. *Let $w \in W$, $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Then, we have*

1. *$H^j(Z(w,\underline{i}), T_{(w,\underline{i})}) = 0$ for all $j \geq 2$.*
2. *If G is simply laced, $H^j(Z(w,\underline{i}), T_{(w,\underline{i})}) = 0$ for all $j \geq 1$.*

Proof. We start by proving (2). We first recall the following isomorphism (see [4, Theorem 3.3.4(b)]):

$$H^j(Z(w,\underline{i}), \phi_w^*\mathcal{L}(w_0, \alpha_{i_r})) \simeq H^j(X(w), \mathcal{L}(w, \alpha_{i_r})) = H^j(w, \alpha_{i_r}) \ for\ all\ j \geq 0.$$

Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Since $f_r : Z(w,\underline{i}) \longrightarrow Z(v,\underline{i}')$ is a smooth fibration with fibre $\mathbb{P}^1$, by using [10, p.288, Corollary 12.9] and [14, p.369, Section 14.6(3)] we have $H^j(Z(w,\underline{i}), f_r^*T_{(v,\underline{i}')}) = H^j(Z(v,\underline{i}'), T_{(v,\underline{i}')})$ for every $j \geq 0$.

By considering the long exact sequence associated to the short exact sequence (rel) and using above arguments, we have the following long exact sequence of B-modules:
$0 \longrightarrow H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(v,\underline{i}'), T_{(v,\underline{i}')}) \longrightarrow H^1(w, \alpha_{i_r}) \longrightarrow$
$H^1(Z(w,\underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(v,\underline{i}), T_{(v,\underline{i}')}) \longrightarrow H^2(w, \alpha_{i_r}) \longrightarrow H^2(Z(w,\underline{i}), T_{(w,\underline{i})}) \longrightarrow$
$H^2(Z(v,\underline{i}'), T_{(v,\underline{i}')}) \longrightarrow H^3(w, \alpha_{i_r}) \longrightarrow \cdots$.
Since G is simply laced, by Lemma 2.6 (2), we have $H^j(w, \alpha_{i_r}) = 0$ for every $j \geq 1$. Thus we have $H^j(Z(w,\underline{i}), T_{(w,\underline{i})}) = H^j(Z(v,\underline{i}'), T_{(v,\underline{i}')})$ for every $j \geq 1$. Now the proof follows by induction on $l(w)$.

Proof of (1) is similar by using Lemma 2.6 (1). $\square$

Note: The long exact sequence associated to the short exact sequence (rel) which is considered in the proof of the Proposition 3.1 will be used frequently in the future. We call this LES.

Proposition 3.1(1) yields $H^2(Z(w,\underline{i}), T_{(w,\underline{i})}) = 0$. Hence, we see that $Z(w,\underline{i})$ has unobstructed deformations. That is, $Z(w,\underline{i})$ admits a smooth versal deformation (see [13, p.273, lines 19-21]).

If in addition G is simply laced, Proposition 3.1(2) yields $H^1(Z(w,\underline{i}), T_{(w,\underline{i})}) = 0$. Using [13, p. 272, Proposition 6.2.10], we see that $Z(w,\underline{i})$ has no deformations. That is, a BSDH variety for a simply laced group G is rigid.

4. Cohomology of the relative tangent bundle on $Z(w,\underline{i})$

In this section, we compute the cohomology groups of the relative tangent bundle on $Z(w,\underline{i})$.

We use the notation as in the previous section. For a B-module V and a character $\mu \in X(T)$, we denote by V_μ, the weight space for the action of T. By the definition, it is the space of all vectors v in V such that, for all $t \in T$, $t \cdot v = \mu(t)v$. We denote by $dim(V_\mu)$ the dimension of the space V_μ.

Given a weight $\lambda \in X(T)$ and a simple root $\gamma \in S$ such that $\langle \lambda, \gamma \rangle \geq 0$, we recall that the γ-string of λ is the set $\{\lambda, \lambda - \gamma, \cdots, \lambda - \langle \lambda, \gamma \rangle \gamma\}$ of weights, which by Lemma 2.1, is the set of weights occuring in $H^0(s_\gamma, \lambda)$.

Recall, the partial order $\leq$ on $X(T)$ given by $\mu \leq \lambda$ if $\lambda - \mu$ is a non-negative integral linear combination of simple roots. We say that $\mu < \lambda$ if in addition $\lambda - \mu$ is non zero.

We begin by proving the following Lemma:

Let R_s (respectively, R_s^-) be the set of short roots (respectively, negative short roots).

Lemma 4.1. *Let $w \in W$, V be a B-module. Then we have*

1. *If there is a character $\lambda_0 \in X(T)$ such that $V_\mu = 0$ unless $\mu \leq \lambda_0$ (respectively, $\mu < \lambda_0$), then $H^0(w, V)_\mu = 0$ unless $\mu \leq \lambda_0$ (respectively, $\mu < \lambda_0$).*
2. *If $V_\mu = 0$ for every $\mu \in X(T) \setminus (R \cup \{0\})$, then $H^0(w, V)_\mu = 0$ for every $\mu \in X(T) \setminus (R \cup \{0\})$.*
3. *If $V_\mu = 0$ for every $\mu \in X(T) \setminus (R_s \cup \{0\})$, then $H^0(w, V)_\mu = 0$ for every $\mu \in X(T) \setminus (R_s \cup \{0\})$.*
4. *If $V_\mu = 0$ for every $\mu \in X(T) \setminus (R_s^- \cup \{0\})$, then $H^0(w, V)_\mu = 0$ for every $\mu \in X(T) \setminus (R_s^- \cup \{0\})$.*

Proof. Proof of (1): Let V be a B-module and $\lambda_0 \in X(T)$ such that $V_\mu = 0$ if $\mu \not\leq \lambda_0$. Proof is by induction on $l(w)$. If $l(w) = 0$ there is nothing to prove. Otherwise, we can choose a $\gamma \in S$ such that $l(s_\gamma w) = l(w) - 1$. Let $u = s_\gamma w$. By *SES*, the B-modules $H^0(s_\gamma, H^0(u, V))$ and $H^0(w, V)$ are isomorphic.

Let $\mu \in X(T)$ be a weight of $H^0(w, V)$ (i.e, $H^0(w,V)_\mu \neq 0$). Then there is an indecomposable B_γ-summand V' of $H^0(u, V)$ such that $H^0(s_\gamma, V')_\mu \neq 0$. By Lemma 2.4, we have $V' = V'' \otimes \mathbb{C}_{\mu'}$ for some irreducible $\widetilde{L}_\gamma$-module V'' and for some character μ' of $\widetilde{B}_\gamma$. By Lemma 2.3, we have $H^0(s_\gamma, V') = V'' \otimes H^0(s_\gamma, \mu')$ and $\langle \mu', \gamma\rangle \geq 0$. Now, let μ'' be the highest weight of V''. Then, $H^0(s_\gamma, V') = H^0(s_\gamma, \mu'') \otimes H^0(s_\gamma, \mu')$. By the description of the weights of $H^0(s_\gamma, \mu'') \otimes H^0(s_\gamma, \mu')$, any weight λ of $H^0(s_\gamma, V')$ is of the form $\lambda = \mu_1 + \mu_2$ where $\mu_1 = \mu'' - a_1\gamma$ and $\mu_2 = \mu' - a_2\gamma$ for some integers $0 \leq a_1 \leq \langle \mu'', \gamma\rangle$, $0 \leq a_2 \leq \langle \mu', \gamma\rangle$. Thus, we have $\lambda = \mu'' + \mu' - (a_1 + a_2)\gamma$.

Hence, any weight λ of $H^0(s_\gamma, V')$ satisfies $\lambda \leq \mu' + \mu''$. In particular, $\mu \leq \mu' + \mu''$. Note that since $\mu' + \mu''$ is the highest weight of $H^0(s_\gamma, V')$, $H^0(u, V)_{\mu'+\mu''} \neq 0$. By induction on $l(w)$, $\mu' + \mu'' \leq \lambda_0$. Hence, we have $\mu \leq \lambda_0$.

Proof of $V_\mu = 0$ unless $\mu < \lambda_0 \Longrightarrow H^0(w, V)_\mu = 0$ unless $\mu < \lambda_0$ is similar.

Proof of (2): Assume that $H^0(w, V)_\mu \neq 0$. We use the same notation as in the proof of (1). We have $H^0(s_\gamma, V') = H^0(s_\gamma, \mu') \otimes H^0(s_\gamma, \mu'')$. Since $V'_{\mu'+\mu''} \neq 0$, by induction on $l(w)$, $\mu' + \mu'' \in R \cup \{0\}$. By the proof of (1), the weights of $H^0(s_\gamma, V')$ are of the form $\mu = \mu' + \mu'' - j\gamma$ for some integer $0 \leq j \leq \langle \mu' + \mu'', \gamma\rangle$. If $\mu' + \mu'' = 0$, then $j = 0$ and so $\mu = \mu' + \mu'' = 0$. Otherwise, $\mu' + \mu''$ is a root, it follows that μ is a root (see [11, p.45, Section 9.4]).

Proof of (3) follows from the proof of (2) because any root in the γ-string of a short root is short.

Proof of (4) follows from (1) (by taking $\lambda_0 = 0$) and (3). ◻

Lemma 4.2. *Let $w \in W$. Then we have, $H^1(w, \mathfrak{b})_\mu = 0$ unless μ is a negative short root.*

Proof. If $l(w) = 0$, we are done. Otherwise, choose $\gamma \in S$ such that $l(s_\gamma w) = l(w) - 1$. Let $u = s_\gamma w$. Then by *SES*, we have

$$0 \longrightarrow H^1(s_\gamma, H^0(u, \mathfrak{b})) \longrightarrow H^1(w, \mathfrak{b}) \longrightarrow H^0(s_\gamma, H^1(u, \mathfrak{b})) \longrightarrow 0$$

By induction on $l(w)$, $H^1(u, \mathfrak{b})_\mu = 0$ unless μ is a negative short root. By Lemma 4.1 (4), $H^0(s_\gamma, H^1(u, \mathfrak{b}))_\mu = 0$ unless μ is a negative short root.

Now, we prove that $H^1(s_\gamma, H^0(u, \mathfrak{b}))_\mu = 0$ unless μ is a negative short root. Assume that $H^1(s_\gamma, H^0(u, \mathfrak{b}))_\mu \neq 0$. Then there exists an indecomposable B_γ-direct summand V_1 of $H^0(u, \mathfrak{b})$ such that $H^1(s_\gamma, V_1)_\mu \neq 0$. By Lemma 2.4, $V_1 = V' \otimes \mathbb{C}_{\mu'}$ for some irreducible $\widetilde{L}_\gamma$-module V' and for some character μ' of $\widetilde{B}_\gamma$. Since $H^1(s_\gamma, V_1) \neq 0$, by Lemma 2.3 we have $\langle \mu', \gamma\rangle \leq -2$ and $H^1(s_\gamma, V_1) = V' \otimes H^0(s_\gamma, s_\gamma \cdot \mu')$. Then any weight μ'' of $H^1(s_\gamma, V_1)$ is in the γ-string from $\mu_1 + \gamma = \mu_1 + \rho - s_\gamma(\rho) = s_\gamma(s_\gamma \cdot \mu_1)$ to $s_\gamma \cdot \mu_1$, where μ_1 is the lowest weight of V_1.

Note that by [16, Lemma 2.6], the evaluation map $ev : H^0(u, \mathfrak{b}) \longrightarrow \mathfrak{b}$ is injective. Hence, if $H^0(u, \mathfrak{b})_{-\gamma} \neq 0$ then $\mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma}$ is an indecomposable B_γ-direct summand of $H^0(u, \mathfrak{b})$ (here h_γ is a basis vector of the zero weight space of $sl_{2,\gamma}$). By Lemma 2.4, we have

$$\mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma} = V \otimes \mathbb{C}_{-\omega_\gamma}$$

where V is the standard 2- dimensional representation of $\widetilde{L}_\gamma$. By Lemma 2.3, we have

$$H^0(s_\gamma, V \otimes \mathbb{C}_{-\omega_\gamma}) = V \otimes H^0(s_\gamma, -\omega_\gamma).$$

Since $\langle -\omega_\gamma, \gamma \rangle = -1$, by Lemma 2.2, $H^1(s_\gamma, \mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma}) = 0$.

Since V_1 is a B-submodule of $\mathfrak{b}$ and $H^1(s_\gamma, V') \neq 0$, by the above arguments, we see that V_1 is not isomorphic to $\mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma}$. In particular, we have $\mu_1 \in R^- \setminus \{-\gamma\}$. Let λ be the lowest weight of V'. Then, we have $\mu_1 = \lambda + \mu'$. Since $\langle \lambda, \gamma \rangle \leq 0$ and $\langle \mu', \gamma \rangle \leq -2$, we have $\langle \mu_1, \gamma \rangle \leq -2$. Further by [11, p.45, Section 9.4], we have $-3 \leq \langle \mu_1, \gamma \rangle$. Then, the γ-string of μ is either $\mu + \gamma$ (if $\langle \mu_1, \gamma \rangle = -2$) or $\mu + \gamma, \mu + 2\gamma$ (if $\langle \mu_1, \gamma \rangle = -3$). In particular, any weight μ'' of $H^1(s_\gamma, V_1)$ satisfies $|\langle \mu'', \gamma \rangle| \leq 1$ and μ'' is a negative short root. In particular, μ is a negative short root.

Hence by the above short exact sequence, we conclude that $H^1(w, \mathfrak{b})_\mu = 0$ unless μ is a negative short root. $\square$

Recall from Section 2 that $h_{\alpha_{i_r}}$ is a basis vector of the zero weight space of $sl_{2,\alpha_{i_r}}$.

Lemma 4.3. *Let $w \in W$ and fix a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$. Then,*

(1) *If there is a $1 \leq j < r - 1$ such that $\alpha_{i_j} = \alpha_{i_r}$, then we have $H^0(w, \alpha_{i_r})_0 = 0$.*
(2) *If $\alpha_{i_j} \neq \alpha_{i_r}$ for all $1 \leq j < r - 1$, then $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is a $B_{\alpha_{i_r}}$-submodule of $H^0(w, \alpha_{i_r})$, and $H^0(w, \alpha_{i_r})_0 = \mathbb{C}.h_{\alpha_{i_r}}$. In particular, $dim(H^0(w, \alpha_{i_r})_0) = 1$.*

Proof. Proof of (1): If there is a $1 \leq j < r - 1$ such that $\alpha_{i_j} = \alpha_{i_r}$, without loss of generality we may assume that there is no k such that $j < k < r - 1$ and $\alpha_{i_k} = \alpha_{i_r}$. Since $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression, there exists a $j < j' \leq r - 1$ such that $\langle \alpha_{i_r}, \alpha_{i_{j'}} \rangle \leq -1$ and $\langle \alpha_{i_r}, \alpha_{i_k} \rangle = 0$ for every k such that $j' < k < r$. By Corollary 2.5, we have the following isomorphism of B-modules:

$$H^0(s_{i_{j'+1}} \cdots s_{i_r}, \alpha_{i_r}) \simeq H^0(s_{i_r}, \alpha_{i_r}) \simeq sl_{2,\alpha_{i_r}}.$$

By *SES*, we have $H^0(s_{i_{j'}} \cdots s_{i_r}, \alpha_{i_r}) \simeq H^0(s_{i_{j'}}, H^0(s_{i_{j'+1}} \cdots s_{i_r}, \alpha_{i_r}))$ as B-modules.

Then,

$$H^0(s_{i_{j'}} \cdots s_{i_r}, \alpha_{i_r}) \simeq H^0(s_{i_{j'}}, H^0(s_{i_r}, \alpha_{i_r})) \simeq \mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}} \oplus \Big(\bigoplus_{m=1}^{-\langle \alpha_{i_r}, \alpha_{i_{j'}} \rangle} \mathbb{C}_{-\alpha_{i_r} - m\alpha_{i_{j'}}} \Big).$$

Since $\langle \alpha_{i_r}, \alpha_{i_k} \rangle \leq 0$ for every $j + 1 \leq k \leq j' - 1$, we conclude that the indecomposable $B_{\alpha_{i_r}}$-summand $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is in the image of the evaluation map

$$ev : H^0(s_{i_{j+1}} \cdots s_{i_{j'-1}}, H^0(s_{i_{j'}} \cdots s_{i_r}, \alpha_{i_r})) \longrightarrow H^0(s_{i_{j'}} \cdots s_{i_r}, \alpha_{i_r}).$$

Since $H^0(s_{i_{j+1}} \cdots s_{i_r}, \alpha_{i_r}) \simeq H^0(s_{i_{j+1}} \cdots s_{i_{j'-1}}, H^0(s_{i_{j'}} \cdots s_{i_r}, \alpha_{i_r}))$, the indecomposable $B_{\alpha_{i_r}}$-module $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is a direct summand of $H^0(s_{i_{j+1}} \cdots s_{i_r}, \alpha_{i_r})$. By similar arguments as in the proof of Lemma 4.2 and using Lemma 2.4, we have $H^0(s_{i_j}, \mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}) = 0$.

Now, let $u_1 = s_{i_1} \cdots s_{i_{j-1}}$ and $u_2 = s_{i_j} \cdots s_{i_r}$. From the above arguments, we see that $H^0(u_2, \alpha_{i_r})_\mu = 0$ unless $\mu < -\alpha_{i_r}$ and $\mu \in R$. By Lemma 4.1, if $H^0(u_1, H^0(u_2, \alpha_{i_r}))_\mu \neq 0$ then $\mu < -\alpha_{i_r}$ and $\mu \in R$. Hence, the zero weight space of $H^0(w, \alpha_{i_r})$ is zero.

Proof of (2): Proof is similar to the proof of (1), for the completeness we will give the proof.
If $\langle \alpha_{i_j}, \alpha_{i_r} \rangle = 0$ for every $1 \leq j \leq r-1$, then by Corollary 2.5, we have $H^0(w, \alpha_{i_r}) = sl_{2,\alpha_{i_r}}$. Hence, (2) holds in this case.

Otherwise, there exists $1 \leq j \leq r-1$ such that $\langle \alpha_{i_j}, \alpha_{i_r} \rangle \neq 0$. Let $1 \leq k \leq r-1$ be the largest integer such that $\langle \alpha_{i_k}, \alpha_{i_r} \rangle \neq 0$. Then by *SES* and Corollary 2.5, we have

$$H^0(w, \alpha_{i_r}) \simeq H^0(s_{i_1}s_{i_2}\cdots s_{i_k}, H^0(s_{i_{k+1}}\cdots s_{i_r}, \alpha_{i_r})) \simeq H^0(s_{i_1}s_{i_2}\cdots s_{i_k}, sl_{2,\alpha_{i_r}}).$$

Since $\langle \alpha_{i_r}, \alpha_{i_k} \rangle \leq -1$, we have

$$H^0(w, \alpha_{i_r}) \simeq H^0(s_{i_1}s_{i_2}\cdots s_{i_{k-1}}, \mathbb{C}.h_{\alpha_{i_r}} \oplus \bigoplus_{m=0}^{-\langle \alpha_{i_k}, \alpha_{i_r} \rangle} \mathbb{C}_{-\alpha_{i_r} - m\alpha_{i_k}}).$$

Since $\alpha_{i_j} \neq \alpha_{i_r}$ for all $1 \leq j < r-1$, we see that $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is an indecomposable $B_{\alpha_{i_r}}$-submodule of $H^0(w, \alpha_{i_r})$. Further, $H^0(w, \alpha_{i_r})_0 = \mathbb{C}.h_{\alpha_{i_r}}$ and so $dim(H^0(w, \alpha_{i_r})_0) = 1$. This completes the proof of the lemma. $\square$

Now onwards we denote by $M_{\geq 0}$ the semi subgroup of $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ generated by the set S of all simple roots .

Lemma 4.4. *Let $w \in W$. Let $\mu \in M_{\geq 0} \setminus \{0\}$ and let $\alpha \in S$. Then, we have*

1. *If $H^0(w, \alpha)_\alpha \neq 0$, then $dim(H^0(w, \alpha)_\alpha) = 1$.*
2. *$H^0(w, \alpha)_\mu \neq 0$ if and only if $\mu = \alpha$ and the evaluation map $ev : H^0(w, \alpha) \longrightarrow \mathbb{C}_\alpha$ is surjective.*

Proof. Proof of (1): Let $w_1 \in W$ be an element of minimal length such that $w_1(\alpha)$ is a dominant weight. Note that if $l(w_1) = 0$, then α is dominant. In particular, G is of rank 1 and $w \in \{id, s_\alpha\}$. Hence $dim(H^0(w, \alpha)_\alpha) = 1$. Otherwise, there exists a $\gamma \in S$ such that $l(w_1 s_\gamma) = l(w_1) - 1$ and $\langle \alpha, \gamma \rangle < 0$. Hence by Lemma 2.1, $\mathbb{C}_\alpha$ is a B-submodule of $H^0(s_\gamma, s_\gamma(\alpha))$. Then $H^0(w, \alpha)$ is a B-submodule of $H^0(w, H^0(s_\gamma, s_\gamma(\alpha)))$. Since $H^0(w, \alpha)_\alpha \neq 0$, by [1, p.110, Theorem 3.3] (see also [5] and [19]) we have $l(ws_\gamma) = l(w) + 1$ (Note that since $\langle \alpha, \gamma \rangle < 0$, the regularity of λ as in [1, p.110, Theorem 3.3] does not play a role). By Lemma 2.2, we have

$$H^0(ws_\gamma, s_\gamma(\alpha)) = H^0(w, H^0(s_\gamma, s_\gamma(\alpha))).$$

Hence $H^0(w, \alpha)$ is a B-submodule of $H^0(ws_\gamma, s_\gamma(\alpha))$. By induction on $l(w_1)$, $H^0(w, \alpha)$ is a B-submodule of $H^0(ww_1^{-1}, w_1(\alpha))$. Since $w_1(\alpha)$ is dominant, $H^0(ww_1^{-1}, w_1(\alpha))$ is a quotient of the B-module $H^0(w_0, w_1(\alpha))$. Further, since the multiplicity of the weight α in $H^0(w_0, w_1(\alpha))$ is 1, the multiplicity of the weight α in $H^0(ww_1^{-1}, w_1(\alpha))$ is at most 1. Hence, we conclude that $dim(H^0(w, \alpha)_\alpha) = 1$.

Proof of (2):

Assume that $H^0(w, \alpha)_\mu \neq 0$. If $l(w) = 0$, there is nothing to prove. Assume $l(w) > 0$. Therefore, we can choose a $\gamma \in S$ such that $l(s_\gamma w) = l(w) - 1$. Let $u = s_\gamma w$. By *SES*, we have $H^0(w, \alpha) = H^0(s_\gamma, H^0(u, \alpha))$.

Since $H^0(w, \alpha)_\mu \neq 0$, there exists an indecomposable B_γ-summand V of $H^0(u, \alpha)$ such that $H^0(s_\gamma, V)_\mu \neq 0$. Let μ' be the highest weight of V. By Lemma 2.4, we have $V = V' \otimes \mathbb{C}_\lambda$ for

some character λ of $\widetilde{B}_\gamma$ and for some irreducible $\widetilde{L}_\gamma$-module V'. Let λ_1 be a highest weight of V'. By similar arguments as in the proof of Lemma 4.1, we have $\lambda_1+\lambda=\mu'$, and $\mu=\mu'-a\gamma$ where $0\leq a\leq\langle\mu',\gamma\rangle$. Therefore, $\mu'=\mu+a\gamma$ for some $a\in\mathbb{Z}_{\geq 0}$ and $H^0(u,\alpha)_{\mu'}\neq 0$. By induction on $l(w)$, $\mu'=\alpha$ and the evaluation map $ev: H^0(u,\alpha)\longrightarrow\mathbb{C}_\alpha$ is surjective. By (1), we see that $ev: H^0(u,\alpha)_\alpha\longrightarrow\mathbb{C}_\alpha$ is an isomorphism. Since $\mu\in M_{\geq 0}\setminus\{0\}$ and $\mu'=\alpha$, we have $a=0$ and hence $\mu=\alpha$. By the above arguments, the restriction of the evaluation map $ev: H^0(w,\alpha)_\alpha\longrightarrow H^0(u,\alpha)_\alpha$ is surjective. Hence, the evaluation map $ev: H^0(w,\alpha)\longrightarrow\mathbb{C}_\alpha$ is surjective.

The other implication is straight forward. ☐

Corollary 4.5. *Let $w\in W$ and fix a reduced expression $w=s_{i_1}s_{i_2}\cdots s_{i_r}$. Let $\mu\in M_{\geq 0}\setminus\{0\}$. Then, we have*

(1) *$H^0(w,\alpha_{i_r})_\mu\neq 0$ if and only if $\mu=\alpha_{i_r}$ and $\langle\alpha_{i_r},\alpha_{i_k}\rangle=0$ for $k=1,2,\cdots,r-1$.*
(2) *In such a case, the evaluation map $ev: H^0(w,\alpha_{i_r})\longrightarrow sl_{2,\alpha_{i_r}}$ is an isomorphism.*

Proof. Proof of (1): Assume that $H^0(w,\alpha_{i_r})_\mu\neq 0$. By Lemma 4.4, we have $\mu=\alpha_{i_r}$ and the evaluation map $ev: H^0(w,\alpha_{i_r})\longrightarrow\mathbb{C}_{\alpha_{i_r}}$ is surjective. We now prove that $\langle\alpha_{i_r},\alpha_{i_k}\rangle=0$ for $k=1,2,\cdots,r-1$. Let $u=s_{i_2}s_{i_3}\cdots s_{i_r}$. Then, we have $l(u)=l(w)-1$. Since the evaluation map $ev: H^0(w,\alpha_{i_r})=H^0(s_{i_1},H^0(u,\alpha_{i_r}))\longrightarrow\mathbb{C}_{\alpha_{i_r}}$ is non zero, the evaluation map $ev: H^0(u,\alpha_{i_r})\longrightarrow\mathbb{C}_{\alpha_{i_r}}$ is non zero, because this evaluation map is the composition of the evaluation maps $H^0(s_{i_1},H^0(u,\alpha))\longrightarrow H^0(u,\alpha)$ and $H^0(u,\alpha)\longrightarrow\mathbb{C}_\alpha$. By induction on $l(w)$, $\langle\alpha_{i_r},\alpha_{i_k}\rangle=0$ for $k=2,\cdots,r-1$. Hence, $w=s_{i_1}s_{i_r}s_{i_2}\cdots s_{i_{r-1}}$ is also a reduced expression for w. In particular, $\alpha_{i_1}\neq\alpha_{i_r}$ and hence $\langle\alpha_{i_1},\alpha_{i_r}\rangle\leq 0$. By Corollary 2.5, we have $H^0(w,\alpha_{i_r})=H^0(s_{i_1}s_{i_r},\alpha_{i_r})$. Note that if $\langle\alpha_{i_1},\alpha_{i_r}\rangle\leq -1$, by Lemma 2.3 we have $H^0(w,\alpha_{i_r})_{\alpha_{i_r}}=0$, which is a contradiction. Thus, we have $\langle\alpha_{i_1},\alpha_{i_r}\rangle=0$. Hence $\langle\alpha_{i_r},\alpha_{i_k}\rangle=0$ for $k=1,2,\ldots,r-1$.

The other implication follows from Corollary 2.5.

Assertion (2) follows from the fact that $H^0(s_{i_r},\alpha_{i_r})$ is the 3-dimensional cyclic B-submodule generated by a weight vector of weight α_{i_r}. ☐

Let $\mathfrak{p}$ be a B-submodule of $\mathfrak{g}$ containing $\mathfrak{b}$.

Lemma 4.6. *Let $w\in W$ and let $\mu\in M_{\geq 0}\setminus\{0\}$. If $H^0(w,\mathfrak{g}/\mathfrak{p})_\mu\neq 0$, then $\mu\in R^+$.*

Proof. If $l(w)=0$, there is nothing to prove. Assume that $l(w)>0$. Then, we can choose $\gamma\in S$ such that $l(s_\gamma w)=l(w)-1$. Let $u=s_\gamma w$. By *SES*, we have $H^0(w,\mathfrak{g}/\mathfrak{p})=H^0(s_\gamma,H^0(u,\mathfrak{g}/\mathfrak{p}))$.

Since $H^0(w,\mathfrak{g}/\mathfrak{p})_\mu\neq 0$, there exists an indecomposable B_γ-summand V of $H^0(u,\mathfrak{g}/\mathfrak{p})$ such that $H^0(s_\gamma,V)_\mu\neq 0$. Let μ' be the highest weight of V. By the same arguments as in the proof of Lemma 4.4, we have $\mu=\mu'-a\gamma$ where $0\leq a\leq\langle\mu',\gamma\rangle$.

Since $l(u)=l(w)-1$ and $V_{\mu'}\neq 0$, by induction on $l(w)$, $\mu'\in R^+$. Hence $\mu'-j\gamma\in R\cup\{0\}$ for every $0\leq j\leq\langle\mu',\gamma\rangle$ (see [11, p.45, Section 9.4]). Since $\mu\in M_{\geq 0}\setminus\{0\}$, we have $\mu\in R^+$. ☐

Proposition 4.7. *Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Fix $1 \leq j \leq r-1$. If $\langle \alpha_{i_j}, \alpha_{i_k}\rangle = 0$ for every $1 \leq k < j$, then the natural map $H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ is surjective.*

Proof. If $l(w) = 0$, there is nothing to prove. Assume $l(w) > 0$ and let $u = s_{i_1}w$. Then, we have $l(u) = l(w) - 1$. By *SES*, we have the evaluation map

$$ev : H^0(w, \mathfrak{g}/\mathfrak{p}) = H^0(s_{i_1}, H^0(u, \mathfrak{g}/\mathfrak{p})) \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{p}).$$

We denote the restriction of the evaluation map ev to $H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ by ev_1.

First we will prove that ev_1 is an isomorphism.

Let v be a non zero vector in $H^0(w, \mathfrak{g}/\mathfrak{p})$ of weight α_{i_j}. Let $H^0(u, \mathfrak{g}/\mathfrak{p})) \simeq \bigoplus_{i=1}^m V_i$ be a decomposition as a sum of indecomposable $B_{\alpha_{i_1}}$-submodules. Since $v \in H^0(s_{i_1}, \bigoplus_{i=1}^m V_i) = \bigoplus_{i=1}^m H^0(s_{i_1}, V_i)$, $v = \sum_{i=1}^m v_i$ where $v_i \in H^0(s_{i_1}, V_i)$ $(1 \leq i \leq m)$, it follows that the weight of v_i is same as the weight of v. Hence, without loss of generality, we may assume that there exists an indecomposable $B_{\alpha_{i_1}}$-summand V of $H^0(u, \mathfrak{g}/\mathfrak{p})$ such that $v \in H^0(s_{i_1}, V)_{\alpha_{i_j}}$. Let μ be the highest weight of V. By the arguments as in the proof of Lemma 4.4, $\mu = \alpha_{i_j} + a\alpha_{i_1}$ for some $a \in \mathbb{Z}_{\geq 0}$. Since $H^0(u, \mathfrak{g}/\mathfrak{p})_\mu \neq 0$, by Lemma 4.6 we see that μ is a positive root. Since either $j = 1$, or $\langle \alpha_{i_j}, \alpha_{i_1}\rangle = 0$, we have $a = 0$. Hence $V = \mathbb{C}.v$. Thus, the map $ev_1 : H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}} \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ is injective. To prove ev_1 is surjective, let v' be a non zero vector in $H^0(u, \mathfrak{g}/\mathfrak{p})$ of weight α_{i_j}. By similar arguments, we may assume that there exists an indecomposable $B_{\alpha_{i_1}}$-summand V' of $H^0(u, \mathfrak{g}/\mathfrak{p})$ containing v'. Let μ' be the highest weight of V'. Then, by the arguments as in the proof of Lemma 4.4, $\mu' = \alpha_{i_j} + a\alpha_{i_1}$ for some $a \in \mathbb{Z}_{\geq 0}$. By the similar arguments as above, we see that $V' = \mathbb{C}.v'$. Hence, we conclude that v' is in the image of ev_1.

In particular, the restriction $ev_2 : H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}}$ of the evaluation map $H^0(w, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{b})$ is an isomorphism.

Now, consider the following commutative diagram of T-modules:

$$\begin{array}{ccc}
H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} & \xrightarrow{f} & H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}} \\
{\scriptstyle ev_2}\downarrow \wr & & {\scriptstyle ev_1}\downarrow \wr \\
H^0(u, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} & \xrightarrow{g} & H^0(u, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}
\end{array}$$

By the induction on $l(w)$, $g : H^0(u, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ is surjective. By the commutativity of the above diagram, it follows that the natural map

$$f : H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$$

is surjective. This completes the proof. $\square$

Corollary 4.8. *Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Fix an integer $j \in \{1, \ldots, r-1\}$ such that for all $1 \leq k < j$, $\langle \alpha_{i_j}, \alpha_{i_k}\rangle = 0$. Then, $H^1(w, \alpha_{i_r})_{\alpha_{i_j}} = 0$.*

Proof. Let $\alpha = \alpha_{i_r}$. Now look at following short exact sequence of B-modules:

$$0 \longrightarrow \mathfrak{g}_\alpha \longrightarrow \mathfrak{g}/\mathfrak{b} \longrightarrow \mathfrak{g}/\mathfrak{p}_\alpha \longrightarrow 0$$

Note that by Theorem 2.7, $H^1(w, \mathfrak{g}/\mathfrak{b}) = 0$. Applying $H^0(w, -)$ to the above short exact sequence of B-modules and taking the α_{i_j} weight spaces, we have the exact sequence of T-modules:

$$0 \longrightarrow H^0(w, \alpha)_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p}_\alpha)_{\alpha_{i_j}} \longrightarrow H^1(w, \alpha)_{\alpha_{i_j}} \longrightarrow 0$$

By Proposition 4.7, we conclude that $H^1(w, \alpha)_{\alpha_{i_j}} = 0$. This completes the proof. □

5. Action of the minimal Parabolic subgroup $P_{\alpha_{i_1}}$ on $Z(w, \underline{i})$

Recall that ϕ_w denotes the birational morphism $Z(w, \underline{i}) \longrightarrow X(w)$. As in Section 2, the composition of inclusion $X(w)$ in G/B with ϕ_w will also be denoted by ϕ_w. Further, we denote the tangent bundle of $Z(w, \underline{i})$ by $T_{(w,\underline{i})}$, where $\underline{i} = (i_1, i_2, \ldots, i_r)$. By using the differential map, we see that $T_{(w,\underline{i})}$ is a subsheaf of $\phi_w^*(T_{G/B})$. Hence $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(Z(w, \underline{i}), \phi_w^*(T_{G/B}))$.

Since the tangent bundle of G/B is the homogeneous vector bundle associated to the representation $\mathfrak{g}/\mathfrak{b}$ of B, we have

$$H^0(Z(w, \underline{i}), \phi_w^*(T_{G/B})) = H^0(w, \mathfrak{g}/\mathfrak{b}).$$

Therefore, $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{b})$.

Denote by $\mathfrak{p}_{\alpha_{i_1}}$, the Lie algebra of the minimal parabolic subgroup $P_{\alpha_{i_1}}$ of G containing B. Note that $\mathfrak{b}$ is contained in $\mathfrak{p}_{\alpha_{i_1}}$.

Lemma 5.1. *Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression $\underline{i}$ for w. Then,*

(1) *There is a non zero homomorphism $f_w : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ of B-modules (which is also a homomorphism of Lie algebras).*
(2) *If $w = w_0$, the homomorphism $f_{w_0} : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ in (1) is injective.*

Proof. Proof of (1): Consider the action of $P_{\alpha_{i_1}}$ on $Z(w, \underline{i})$ induced by the following left action of $P_{\alpha_{i_1}}$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$:
Let $p \in P_{\alpha_{i_1}}$ and $x = (p_1, p_2, \cdots, p_r) \in P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ then $p.x := (pp_1, p_2, \cdots, p_r)$.

Clearly, this action is non trivial. Hence, there is a non trivial homomorphism

$$\psi_w : P_{\alpha_{i_1}} \longrightarrow Aut^0(Z(w, \underline{i}))$$

of algebraic groups. Consider the action of B on $P_{\alpha_{i_1}}$ by conjugation and the action of B on $Aut^0(Z(w, \underline{i}))$ via ψ_w. Note that ψ_w is B-equivariant.

By [17, Theorem 3.7], $Aut^0(Z(w, \underline{i}))$ is an algebraic group and

$$Lie(Aut^0(Z(w, \underline{i}))) = H^0(Z(w, \underline{i}), T_{(w,\underline{i})}).$$

Then, the induced homomorphism

$$f_w : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of B-modules (homomorphism of Lie algebras) is non zero.

Proof of (2): Since $f_w : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ is a non zero homomorphism of B-modules (homomorphism of Lie algebras), $f_w(\mathfrak{p}_{\alpha_{i_1}})$ contains a B-stable line L. Let μ be the character of B such that $b.v = \mu(b).v$ for all $b \in B$ and for all $v \in L$. That is, L is the one-dimensional space generated by a lowest weight vector of weight μ.

Since $w = w_0$, $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $H^0(G/B, T_{G/B})$. By Bott's theorem [2, Theorem VII] we have $H^0(G/B, T_{G/B}) = \mathfrak{g}$. Hence $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $\mathfrak{g}$. Since there is a unique B-stable one-dimensional subspace L of $\mathfrak{g}$ and the character of B is $-\alpha_0$, we conclude that $\mu = -\alpha_0$ and $L = \mathfrak{g}_{-\alpha_0} \subset f_{w_0}(\mathfrak{p}_{\alpha_{i_1}})$. By the similar arguments, the unique B-stable one-dimensional subspace in $\mathfrak{p}_{\alpha_{i_1}}$ is $\mathfrak{g}_{-\alpha_0}$. Hence f_{w_0} is injective (otherwise $Ker(f_{w_0}) \neq 0$ and hence the unique B-stable line $\mathfrak{g}_{-\alpha_0}$ is a subspace of $Ker(f_{w_0})$, which is a contradiction). $\square$

Corollary 5.2.

(1) $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ *is a Lie subalgebra of* $\mathfrak{g}$.
(2) *Any Borel subalgebra of* $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ *is isomorphic to* $\mathfrak{b}$.
(3) *Any maximal toral subalgebra of* $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ *is isomorphic to* $\mathfrak{h}$.

Proof. Proof of (1): By Lemma 5.1(2), $\mathfrak{b}$ is a Lie subalgebra of $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$. Since $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $\mathfrak{g}$, for any $Y \in H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ and for any $X \in \mathfrak{b}$ the Lie bracket $[X, Y]$ in $\mathfrak{g}$ is same as the Lie bracket in $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$. It remains to prove that for every $\alpha, \beta \in R^+$ such that α, β are weights of $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$, the Lie bracket $[x_\beta, x_\alpha]$ in $\mathfrak{g}$ is same as the Lie bracket in $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$.

Note that the Lie subalgebra of $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ generated by $\mathfrak{g}_\beta \cap H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ for $\beta \in R^+$ is same as the Lie subalgebra generated by $\mathfrak{g}_\alpha \cap H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ for $\alpha \in S$. Hence it is enough to prove that for every $\beta \in R^+$ and $\alpha \in S$ such that α, β are weights of $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$, the Lie bracket $[x_\beta, x_\alpha]$ in $\mathfrak{g}$ is same as the Lie bracket in $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$.

Let $[-,-]'$ be the Lie bracket in $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$. For $\beta \in R^+, \alpha \in S$, by Jacobi identity we have

$$[x_{-\beta}, [x_\beta, x_\alpha]']' = [[x_{-\beta}, x_\beta]', x_\alpha]' + [x_\beta, [x_{-\beta}, x_\alpha]']'.$$

Since $x_{-\beta} \in \mathfrak{b}$ and $\mathfrak{b}$ is a Lie subalgebra of $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ and $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $\mathfrak{g}$, we have

$$[x_{-\beta}, x_\beta]' = [x_{-\beta}, x_\beta] \textit{ and } [x_{-\beta}, x_\alpha]' = [x_{-\beta}, x_\alpha].$$

Hence, we have

$$[x_{-\beta}, [x_\beta, x_\alpha]']' = [[x_{-\beta}, x_\beta], x_\alpha]' + [x_\beta, [x_{-\beta}, x_\alpha]]'. \tag{5.1}$$

Note that $[x_{-\beta}, x_\beta], [x_{-\beta}, x_\alpha] \in \mathfrak{b}$. Therefore, by (5.1) and Jacobi identity we have

$$[x_{-\beta}, [x_\beta, x_\alpha]']' = [[x_{-\beta}, x_\beta], x_\alpha] + [x_\beta, [x_{-\beta}, x_\alpha]] = [x_{-\beta}, [x_\beta, x_\alpha]].$$

Since $x_{-\beta} \in \mathfrak{b}$, we have $[x_{-\beta}, [x_\beta, x_\alpha]']' = [x_{-\beta}, [x_\beta, x_\alpha]']$. Hence, we have

$$[x_{-\beta}, [x_\beta, x_\alpha]'] = [x_{-\beta}, [x_\beta, x_\alpha]]. \tag{5.2}$$

If $[x_\beta, x_\alpha] = 0$, then $\alpha + \beta \notin R$. In particular, $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})_{\alpha+\beta} = 0$. Then, we have $[x_\beta, x_\alpha]' = 0$.

If $[x_\beta, x_\alpha] \neq 0$, then $\alpha + \beta \in R$.

$$[x_{-\beta}, [x_\beta, x_\alpha]] = [x_\beta, [x_{-\beta}, x_\alpha]] - h_\beta \cdot x_\alpha.$$

If $[x_{-\beta}, x_\alpha] = 0$ and $h_\beta \cdot x_\alpha = 0$, then α, β are orthogonal and $\beta - \alpha \notin R$. Hence, we have $\alpha + \beta \notin R$. This contradicts the assumption. Hence, we have $[x_\beta, x_\alpha]' = c_1 x_{\alpha+\beta}$ and $[x_\beta, x_\alpha] = c_2 x_{\alpha+\beta}$, with $c_2 \neq 0$. Therefore, by (5.2) it follows that $c_1 = c_2$ and $[x_\beta, x_\alpha]' = [x_\beta, x_\alpha]$. Hence $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$.

Proof of (2): By (1), $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$. Now we claim that $\mathfrak{b}$ is a Borel subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$. Otherwise, there exists a Borel subalgebra $\mathfrak{b}'$ of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ properly containing $\mathfrak{b}$. Since $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$, we see that $\mathfrak{g}_\alpha \subset \mathfrak{b}'$ for some simple root α. Since $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is a Lie subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, the simple Lie algebra $sl_{2,\alpha}$ is a Lie subalgebra of $\mathfrak{b}'$, which is a contradiction to the solvability of $\mathfrak{b}'$. Hence $\mathfrak{b}$ is a Borel subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$. Since any two Borel subalgebras of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ are conjugate (see [11, p.84, Theorem 16.4]), we conclude (2).

Proof of (3): Since any two maximal toral subalgebras of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ are conjugate (see [11, p.84, Corollary 16.4]), the proof follows from (2). $\square$

Let $w \in W$, let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression $\underline{i}$ of w. Fix a reduced expression $w_0 = s_{j_1} s_{j_2} \cdots s_{j_r} s_{j_{r+1}} \cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \ldots j_N)$ and $\underline{i} = (j_1, j_2, \ldots j_r)$. Let $v = s_{j_{r+1}} s_{j_{r+2}} \cdots s_{j_N}$ and $\underline{j}' = (j_{r+1}, \ldots, j_N)$.

Since the $Z(v, \underline{j}')$-fibration $Z(w_0, \underline{j}) \longrightarrow Z(w, \underline{i})$ is $P_{\alpha_{i_1}}$ equivariant, it follows that

$$H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

is a homomorphism of $P_{\alpha_{i_1}}$-modules. Hence, it is a homomorphism of $\mathfrak{p}_{\alpha_{i_1}}$-modules. Thus, the restriction of this map to $\mathfrak{p}_{\alpha_{i_1}}$ is the same as the map induced by the action of $P_{\alpha_{i_1}}$ on $Z(w, \underline{i})$.

Note that since $f_{w_0} : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is injective (see Lemma 5.1(2)), we identify $\mathfrak{p}_{\alpha_{i_1}}$ as a Lie subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$.

Hence, we have the following commutative diagram of $P_{\alpha_{i_1}}$-modules:

$$\begin{tikzcd} \mathfrak{p}_{\alpha_{i_1}} \arrow[r, hook] \arrow[dr, "f_w"'] & H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \arrow[d] \\ & H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \end{tikzcd}$$

Further, the maps in the above diagram are homomorphisms of Lie algebras.

For simplicity of notation, we denote both the natural map

$$H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

and its restriction to $\mathfrak{p}_{\alpha_{i_1}}$ by f_w.

Let $d(w)$ be the number of distinct i_j's in $\underline{i} = (i_1, i_2, \ldots, i_r)$ (i.e, the number of distinct simple reflections s_{i_j}'s appearing in the reduced expression $\underline{i}$ of w). Let $\leq$ be the Bruhat-Chevalley ordering on W. Note that $d(w)$ is equal to the number of distinct Schubert curves in $X(w)$. That is, $d(w)$ is equal to the number of distinct $j \in \{1, 2, \ldots, n\}$ such that $s_j \leq w$. In particular, it is independent of the choice of the reduced expression $\underline{i}$ of w. Further, we also note that $d(w_0) = n$.

Now, we prove the following Lemma:

Lemma 5.3.

(1) *The dimension of the zero weight space* $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$ *is at most* $d(w)$.
(2) *In particular,* $dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq rank(G)$.

Proof. Consider the following short exact sequence of B-modules:

$$0 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b} \longrightarrow 0$$

By applying $H^0(w, -)$ to the above short exact sequence, we have the following exact sequence of B-modules:

$$0 \longrightarrow H^0(w, \mathfrak{b}) \longrightarrow H^0(w, \mathfrak{g}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^1(w, \mathfrak{b}) \longrightarrow 0$$

(Note that $H^1(w, \mathfrak{g}) = 0$ (see [16, Lemma 2.5(2)]))

By Lemma 4.2, we have $H^1(w, \mathfrak{b})_0 = 0$. Since $H^0(w, \mathfrak{g}) = \mathfrak{g}$, by taking the zero weight space to the above exact sequence we have the following short exact sequence of T-modules;

$$0 \longrightarrow H^0(w, \mathfrak{b})_0 \longrightarrow \mathfrak{h} \xrightarrow{\phi} H^0(w, \mathfrak{g}/\mathfrak{b})_0 \longrightarrow 0$$

Claim: $dim(H^0(w, \mathfrak{b})_0) = rank(G) - d(w)$.

We use the similar arguments as in the proof of Lemma 4.2 and Lemma 4.3 to prove the claim.

Let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression $\underline{i}$ of w. Since S is a basis for the complex vector space $\mathfrak{h}$, for every $1 \leq j \leq n$ there exists a $h(\alpha_j) \in \mathfrak{h}$ such that $\alpha_i(h(\alpha_j)) = \delta_{i,j}$ for $1 \leq i \leq n$. First note that for every $i \neq j$, the one-dimensional subspace $\mathbb{C}h(\alpha_j)$ of $\mathfrak{h}$ is an indecomposable B_{α_i}-direct summand of $\mathfrak{b}$. Therefore, the image of the evaluation map $ev : H^0(w, \mathfrak{b}) \longrightarrow \mathfrak{b}$ contains $h(\alpha_j)$ for every $1 \leq j \leq n$ such that $s_j \not\leq w$. Let $1 \leq k \leq n$ such that $s_k \leq w$. Let $1 \leq j_0 \leq r$ be the largest integer such that $i_{j_0} = k$, let $u = s_{i_{j_0+1}} \cdots s_{i_r}$. Note that since $\alpha_{i_j}(h(\alpha_k)) = 0$ for $j_0 + 1 \leq j \leq r$, $\mathbb{C}h_{\alpha_k}$ is contained in the image of the evaluation map $ev : H^0(u, \mathfrak{b}) \longrightarrow \mathfrak{b}$. Therefore, $\mathbb{C}h(\alpha_k) \oplus \mathbb{C}_{-\alpha_k}$ is an indecomposable B_{α_k}-direct summand of $H^0(u, \mathfrak{b})$ (see [16, Lemma 3.3]).

Further, by Lemma 2.4

$$\mathbb{C}h(\alpha_k) \oplus \mathbb{C}_{-\alpha_k} = V \otimes \mathbb{C}_{-\omega_k}$$

where V is the standard 2- dimensional representation of $\widetilde{L}_{\alpha_k}$. Therefore, by Lemma 2.3 and Lemma 2.2, $H^0(s_{i_{j_0}}, \mathbb{C}h(\alpha_k) \oplus \mathbb{C}_{-\alpha_k}) = 0$.

Let $v = s_{i_{j_0}} u$. By *SES*, we conclude that $H^0(v, \mathfrak{b})_0 = \bigoplus_{\{i : s_i \not\leq v\}} \mathbb{C}h(\alpha_i)$. In view of [16, Lemma 2.6], $H^0(w, \mathfrak{b})_0 = \bigoplus_{\{i : s_i \not\leq w\}} \mathbb{C}h(\alpha_i)$.

Then by the above claim and the short exact sequence, we have

$$dim(H^0(w, \mathfrak{g}/\mathfrak{b})_0) = d(w).$$

Since $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{b})$, we have

$$dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq d(w).$$

□

6. The B-module of the global sections of the tangent bundle on $Z(w, \underline{i})$

In this section, we study the B-module of the global sections of the tangent bundle on $Z(w, \underline{i})$. In particular, we prove that the dimension of the zero weight space of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is equal to $d(w)$, the number of Schubert curves in $X(w)$. We also prove that $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ contains a Lie subalgebra $\mathfrak{b}'$ isomorphic to $\mathfrak{b}$ if and only if $w^{-1}(\alpha_0) < 0$.

We use the notation as in the previous section.

Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Let $supp(w) := \{j \in \{1, 2, \ldots, n\} : s_j \leq w\}$, the support of w. Note that $d(w) = |supp(w)|$.

We have the following proposition:

Proposition 6.1.

(1) *$\{f_w(h_{\alpha_{i_j}}) : j \in supp(w)\}$ forms a basis of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$.*
(2) *In particular, $dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) = d(w)$.*
(3) *The image $f_w(\mathfrak{h})$ is a maximal toral subalgebra of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$.*

Proof. If $w = w_0$, then by Lemma 5.1(2), f_{w_0} is injective and by Corollary 5.2,

$$dim(H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}))_0) = rank(G) = d(w_0).$$

Hence, $\{f_{w_0}(h_{\alpha_{i_j}}) : j \in supp(w_0)\}$ forms a basis of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})_0$.

Otherwise, choose a reduced expression $w_0 = s_{j_1}s_{j_2}\ldots s_{j_N}$ of w_0 such that $(j_1, j_2, \cdots, j_r) = \underline{i}$. Let $v = s_{j_1}s_{j_2}\cdots s_{j_{r+1}}$ and $\underline{i}' = (j_1, \ldots j_r, j_{r+1})$. Note that $l(v) = l(w) + 1$. By descending induction on $l(w)$, $\{f_v(h_{\alpha_{i_j}}) : j \in supp(v)\}$ forms a basis of $H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_0$ and

$$dim(H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}))_0) = d(v).$$

Note that by Lemma 5.3, $dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq d(w)$. By using *LES* and Lemma 2.6, we have the following exact sequence of B-modules:

$0 \longrightarrow H^0(v, \alpha_{i_{r+1}}) \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(v, \alpha_{i_{r+1}}) \longrightarrow H^1(Z(v, \underline{i}'), T_{(v,\underline{i}')}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow 0.$

By taking the zero weight spaces, we have the following exact sequence of T-modules:

$$0 \longrightarrow H^0(v, \alpha_{i_{r+1}})_0 \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_0 \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0 \longrightarrow H^1(v, \alpha_{i_{r+1}})_0 \cdots$$

First assume that there exists $1 \leq j \leq r$ such that $\alpha_{i_j} = \alpha_{i_{r+1}}$, so that $d(v) = d(w)$. By Lemma 4.3, we have $H^0(v, \alpha_{i_{r+1}})_0 = 0$. Hence

$$d(v) = dim(H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}))_0) \leq dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq d(w).$$

Since $d(w) = d(v)$, we have

$$dim(H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_0) = d(v) = d(w) = dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0).$$

Hence, by the above exact sequence, we conclude that $\{f_w(h_{\alpha_{i_j}}) : j \in supp(w)\}$ forms a basis of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$.

Otherwise $d(w) = d(v) - 1$ and by Lemma 4.3(2), we see that $H^0(v, \alpha_{i_{r+1}})_0 = \mathbb{C}.h_{\alpha_{i_{r+1}}}$. By using the above exact sequence, we see that

$$dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \geq d(v) - 1.$$

Since $dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq d(w)$, we conclude that

$$dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) = d(w)$$

and hence $\{f_w(h_{\alpha_{i_j}}) : j \in supp(w)\}$ forms a basis of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$. This completes the proof of (1) and (2).

Proof of (3):

By Lemma 5.1(2), $f_{w_0} : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$ is an injective homomorphism of Lie algebras. By Corollary 5.2(1), $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$ is a Lie subalgebra of $\mathfrak{g}$. Hence, we have

$$H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})_0 = \mathfrak{h}.$$

Let $u = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Note that $l(u) = l(w) - 1$.

Consider the homomorphism $f : H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')})$ of Lie algebras induced by the $\mathbb{P}^1$-fibration $f_r : Z(w, \underline{i}) \longrightarrow Z(u, \underline{i}')$ as in Section 2. By *LES*, $Ker(f) = H^0(w, \alpha_{i_r})$.

Note that by Lemma 4.1(1),

$$H^0(w, \alpha_{i_r})_\mu = 0 \textit{ unless } \mu \leq \alpha_{i_r}. \tag{6.1}$$

Case 1: If $s_{i_r} \leq u$, then by Lemma 4.3(1), $H^0(w, \alpha_{i_r})_0 = 0$. Hence by Corollary 4.5 and Lemma 4.1(2), we conclude that $H^0(w, \alpha_{i_r})_\mu = 0$ unless $\mu \in R^-$. Since for every $\beta \in R^+$, $ad(x_{-\beta})^r = 0$ in $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ for some $r \in \mathbb{N}$ (since for every positive root α, there is a $r \in \mathbb{N}$ such that $\alpha + k\beta \notin R$ for all $k \geq r$), we conclude that every element of $H^0(w, \alpha_{i_r}) \subseteq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is nilpotent.

Case 2: Assume that $s_{i_r} \not\leq u$.

Sub case (a): If $\langle \alpha_{i_j}, \alpha_{i_r} \rangle \neq 0$ for some $1 \leq j \leq r - 1$, then by Corollary 4.5(1), we have $H^0(w, \alpha_{i_r})_{\alpha_{i_r}} = 0$. Hence by (6.1), we have $H^0(w, \alpha_{i_r})_\mu = 0$ unless $\mu \leq 0$. Therefore, again by Lemma 4.1(2) $H^0(w, \alpha_{i_r})_\mu = 0$ unless $\mu \in R^- \cup \{0\}$. Further, by Lemma 4.3, $H^0(w, \alpha_{i_r})_0 = \mathbb{C}.h_{\alpha_{i_r}}$. Hence, a maximal toral subalgebra of $H^0(w, \alpha_{i_r}) \subseteq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ lies in $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ and so it is one-dimensional.

Sub case (b): If $\langle \alpha_{i_j}, \alpha_{i_r} \rangle = 0$ for all $1 \leq j \leq r-1$, then by Corollary 2.5, we have

$$H^0(w, \alpha_{i_r}) \simeq sl_{2,\alpha_{i_r}}.$$

Hence, any maximal toral subalgebra of the ideal $H^0(w, \alpha_{i_r}) \subseteq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ lies in $sl_{2,\alpha_{i_r}}$ and so it is one-dimensional.

Hence, it follows that

$$f_w(\mathfrak{h}) \cap Ker(f) = Ker(f)_0 = H^0(w, \alpha_{i_r})_0$$

is a maximal toral subalgebra of $Ker(f)$ and its dimension is at most one.

By induction on $l(w)$ and by (1), $f_u(\mathfrak{h}) = H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')})_0$ is a maximal toral subalgebra of $H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')})$.

Now, consider the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \arrow[d, "f_w"'] \arrow[dr, "f_u"] & \\
H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \arrow[r, "f"] & H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')})
\end{tikzcd}$$

Note that by commutativity of the above diagram and by (1), it follows that $f_w(\mathfrak{h})$ is an extension of $f_u(\mathfrak{h})$ and $f_w(\mathfrak{h}) \cap Ker(f)$. Thus, we conclude that $f_w(\mathfrak{h}) = H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$ is a maximal toral subalgebra of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$. This completes the proof of the proposition.

$\square$

Consider the restriction of the homomorphism $f_w : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$(as in Lemma 5.1) to $\mathfrak{b}$ and denote it also by f_w.

Lemma 6.2. *The homomorphism* $f_w : \mathfrak{b} \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ *is injective if and only if* $w^{-1}(\alpha_0) < 0$.

Proof. Assume that f_w is injective. Since $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{b})$, we have $H^0(w, \mathfrak{g}/\mathfrak{b})_{-\alpha_0} \neq 0$.

Recall from the proof the Lemma 5.3, the following exact sequence of B-modules:

$$0 \longrightarrow H^0(w, \mathfrak{b}) \longrightarrow \mathfrak{g} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^1(w, \mathfrak{b}) \longrightarrow 0$$

Note that if G is simply laced, by [16, Lemma 3.4] $H^1(w, \mathfrak{b}) = 0$. If G is non simply laced, since $-\alpha_0$ is a long root by [16, Lemma 4.8(2)], we have $H^1(w, \mathfrak{b})_{-\alpha_0} = 0$. Hence, we have the following short exact sequence of T-modules:

$$0 \longrightarrow H^0(w, \mathfrak{b})_{-\alpha_0} \longrightarrow \mathfrak{g}_{-\alpha_0} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b})_{-\alpha_0} \longrightarrow 0$$

Since $dim(\mathfrak{g}_{-\alpha_0}) = 1$, $H^0(w, \mathfrak{b})_{-\alpha_0} = 0$. Hence, we have $w^{-1}(\alpha_0) < 0$.

Now we prove the converse.

Let $\psi_w : B \longrightarrow Aut^0(Z(w,\underline{i}))$ be the homomorphism of algebraic groups induced by the action of B on $Z(w,\underline{i})$(as in the proof of Lemma 5.1). Let K be the kernel of ψ_w. Since

$$BwB/B = \prod_{\beta \in R^+(w)} U_{-\beta}wB/B$$

(see [14, Section 13.1]) and $w^{-1}(\alpha_0) < 0$, we have

$$U_{-\alpha_0}wB/B \neq wB/B.$$

Since the desingularization map $\phi_w : Z(w,\underline{i}) \longrightarrow X(w)$ is B-equivariant and the restriction of ϕ_w to an open subset is an isomorphism onto BwB/B, we have $U_{-\alpha_0} \cap K = \{e\}$, where e is identity element in B.

Recall that $f_w : \mathfrak{b} \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ is the homomorphism of Lie algebras induced by ψ_w. Since $U_{-\alpha_0} \cap K = \{e\}$, we have

$$(Ker(f_w))_{-\alpha_0} = 0.$$

Since $Ker(f_w)$ is a B-submodule of $\mathfrak{b}$ and $\mathfrak{b}$ has a unique B-stable line $\mathfrak{g}_{-\alpha_0}$, we have $Ker(f_w) = 0$. Hence f_w is injective. □

The following proposition describes the set of all positive roots occurring as a weight in $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$.

Proposition 6.3. *Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Let $\mu \in M_{\geq 0} \setminus \{0\}$. Then, we have*

1. *$H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\mu \neq 0$ if and only if there exists an integer $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$, and $\mu = \alpha_{i_j}$.*
2. *Fix $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$. Then, we have $dim(H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_{\alpha_{i_j}}) = 1$ and $sl_{2,\alpha_{i_j}}$ is a $B_{\alpha_{i_j}}$-submodule of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$.*

Proof. Proof of (1): Assume that $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\mu \neq 0$. Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and let $i' = (i_1, i_2, \ldots, i_{r-1})$. By using LES and Lemma 2.6, we have the following exact sequence of B-modules:
$0 \longrightarrow H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(v,\underline{i}'), T_{(v,\underline{i}')}) \longrightarrow H^1(w, \alpha_{i_r}) \longrightarrow$
$H^1(Z(w,\underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(v,\underline{i}'), T_{(v,\underline{i}')}) \longrightarrow 0.$

Since $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\mu \neq 0$, either $H^0(w, \alpha_{i_r})_\mu \neq 0$ or $H^0(Z(v,\underline{i}'), T_{(v,\underline{i}')})_\mu \neq 0$.

Now, if $H^0(w, \alpha_{i_r})_\mu \neq 0$, then by Corollary 4.5, we are done.

Otherwise, we have $H^0(Z(v,\underline{i}'), T_{(v,\underline{i}')})_\mu \neq 0$. Then by the induction on $l(w)$, there exists $1 \leq j \leq r-1$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$ and $\mu = \alpha_{i_j}$.

We now prove the other implication:

Let $1 \leq j \leq r$ be such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$.

If $j = r$, then $\langle \alpha_{i_k}, \alpha_{i_r} \rangle = 0$ for all $1 \leq k \leq r-1$. By Corollary 4.5, we have

$$H^0(w, \alpha_{i_r})_{\alpha_{i_r}} \neq 0.$$

Hence, we conclude that

$$H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_{\alpha_{i_r}} \neq 0.$$

Otherwise, by Corollary 4.5, we have $H^0(w,\alpha_{i_r})_{\alpha_{i_j}} = 0$ and by Corollary 4.8, we have $H^1(w,\alpha_{i_r})_{\alpha_{i_j}} = 0$. By the above exact sequence, we get

$$H^0(Z(w,\underline{i}),T_{(w,i)})_{\alpha_{i_j}} \simeq H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})_{\alpha_{i_j}}.$$

Now the proof follows by induction on $l(w)$.

Proof of (2): Fix $1 \leq j \leq r$. Assume that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$. Then, by (1), we have $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_{\alpha_{i_j}} \neq 0$.

Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $i' = (i_1, i_2, \ldots, i_{r-1})$.

If $j = r$, then by Corollary 4.5 we have $H^0(w,\alpha_{i_r}) \simeq sl_{2,\alpha_{i_r}}$. Also, by using (1), we see that $H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})_{\alpha_{i_r}} = 0$. Hence, by the above exact sequence, we conclude that $dim(H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_{\alpha_{i_r}}) = 1$ and $sl_{2,\alpha_{i_r}}$ is a $B_{\alpha_{i_r}}$-submodule of $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$.

On the other hand, if $j \neq r$ then by induction on $l(w)$,

$$dim(H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})_{\alpha_{i_j}}) = 1$$

and $sl_{2,\alpha_{i_j}}$ is a $B_{\alpha_{i_j}}$-submodule of $H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})$. Note that by Corollary 4.5, we have $H^0(w,\alpha_{i_r})_{\alpha_{i_j}} = 0$. Also, by Corollary 4.8, we have $H^1(w,\alpha_{i_r})_{\alpha_{i_j}} = 0$. Hence, by the above exact sequence, we see that

$$H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_{\alpha_{i_j}} \simeq H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})_{\alpha_{i_j}}$$

and $dim(H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_{\alpha_{i_j}}) = 1$.

Further, since $sl_{2,\alpha_{i_j}}$ is a cyclic $B_{\alpha_{i_j}}$-module generated by $x_{\alpha_{i_j}}$, it follows that $x_{\alpha_{i_j}}$ is in the image of the map $H^0(Z(w,\underline{i}),T_{(w,\underline{i})}) \longrightarrow H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})$. Thus, we conclude that $sl_{2,\alpha_{i_j}}$ is a $B_{\alpha_{i_j}}$-submodule of $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$. $\square$

Proposition 6.4. *Let $w \in W$ and $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression $\underline{i}$ of w. Then, $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ contains a Lie subalgebra $\mathfrak{b}'$ isomorphic to $\mathfrak{b}$ if and only if $w^{-1}(\alpha_0) < 0$.*

Proof. Recall from the proof of Lemma 6.2, $\psi_w : B \longrightarrow Aut^0(Z(w,\underline{i}))$ is the homomorphism of algebraic groups induced by the action of B on $Z(w,\underline{i})$ and $f_w : \mathfrak{b} \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is the induced homomorphism of Lie algebras.

Assume that $\mathfrak{b}'$ is a Lie subalgebra of $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ which is isomorphic to $\mathfrak{b}$, then there exists a closed subgroup B' of $Aut^0(Z(w,\underline{i}))$ such that B' is isomorphic to B and $Lie(B') = \mathfrak{b}'$.

Fix an isomorphism $g : B \longrightarrow B'$. Then, $g(T)(\simeq T)$ is a maximal torus in B'. Hence, we have

$$rank(Aut^0(Z(w,\underline{i}))) \geq dim(T).$$

By Proposition 6.1(3), $f_w(\mathfrak{h})$ is a maximal toral subalgebra of $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$. Hence, $\psi_w(T)$ is a maximal torus in $Aut^0(Z(w,\underline{i}))$. Thus, the restriction $\psi_w|_T : T \longrightarrow Aut^0(Z(w,\underline{i}))$ is injective.

Let T' be a maximal torus of B'. Since any two maximal tori in $Aut^0(Z(w,\underline{i}))$ are conjugate, there exists a $\sigma \in Aut^0(Z(w,\underline{i}))$ such that $T = \sigma T'\sigma^{-1}$. Now, let $B'' := \sigma B'\sigma^{-1}$. Then, we have $T \subset B''$. Since $Lie(B'')$ is a T-stable Lie subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$, by Proposition 6.3 we have

$$Lie(B'') = \mathfrak{h} \oplus \bigoplus_{\beta\in R'} \mathfrak{g}_\beta \oplus \bigoplus_{\alpha\in S'} \mathfrak{g}_\alpha$$

for some subset R' of R^- and for some subset S' of S.

Fix $\alpha \in S'$, Then, we have $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\alpha \neq 0$. Hence by Proposition 6.3, we have $dim(H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\alpha) = 1$. Thus, the homomorphism $f_w : \mathfrak{b} \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ extends to $\widetilde{f_w} : \mathfrak{p}_\alpha \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ as T-modules such that $\widetilde{f_w}(\mathfrak{g}_\alpha) \neq 0$. Let $\mathfrak{l}_\alpha \subseteq \mathfrak{p}_\alpha$ be the Lie algebra of L_α. Consider the restriction $(f_w)_\alpha$ of $\widetilde{f_w}$ to $\mathfrak{l}_\alpha$. Clearly, $(f_w)_\alpha$ is injective homomorphism of Lie algebras. Let n_α be a representative of the simple reflection s_α in $N_G(T)$, let $(\psi_w)_\alpha : \widetilde{L}_\alpha \longrightarrow Aut^0(Z(w,\underline{i}))$ be the homomorphism of algebraic groups induced by $f_{w\alpha}$, where $\widetilde{L}_\alpha$ is a simply connected covering of L_α. Since $(f_w)_\alpha$ is injective, $\widetilde{n}_\alpha \notin Ker((\psi_w)_\alpha)$, where $\widetilde{n}_\alpha$ is a lift of n_α in $\widetilde{L}_\alpha$. Note that $(\psi_w)_\alpha(n_\alpha)$ normalizes T and hence $Ad((\psi_w)_\alpha(n_\alpha))(\mathfrak{h}) = \mathfrak{h}$.

Since $Lie(B'')$ is solvable Lie subalgebra and $\mathfrak{g}_\alpha \subseteq Lie(B'')$, $\mathfrak{g}_{-\alpha} \nsubseteq Lie(B'')$ (otherwise, $sl_{2,\alpha}$ would be Lie subalgebra of $Lie(B'')$). Hence, we have $R' \cap (-S') = \emptyset$.

Note that by Proposition 6.3, if $\alpha \in S'$, then $\alpha = \alpha_{i_j}$ for some integer $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k}\rangle = 0$ for all $1 \leq k \leq j-1$. Hence, the elements in $\{s_\alpha : \alpha \in S'\}$ commute with each other. Thus, $(\prod_{\alpha\in S'} s_\alpha)(\beta) = -\beta$ for every $\beta \in S'$. Further, since $R' \cap (-S') = \emptyset$, we have $(\prod_{\alpha\in S'} s_\alpha)(R') \subseteq R^-$. Let $n = \prod_{\alpha\in S'}(\psi_w)_\alpha(\widetilde{n}_\alpha)$, where the product is taken in some ordering. Hence

$$Lie(nB''n^{-1}) = \mathfrak{h} \oplus \bigoplus_{\beta\in R''} \mathfrak{g}_\beta \oplus \bigoplus_{\gamma\in S'} \mathfrak{g}_\gamma,$$

where $R'' = (\prod_{\alpha\in S'} s_\alpha)(R')$. Note that for each $\alpha \in S'$, $s_\alpha(R') \cap (-S') = \emptyset$. Hence $R'' \cap (-S') = \emptyset$. Then, $Lie(nB''n^{-1}) \subseteq \mathfrak{b}$. Since $dim(\mathfrak{b}) = dim(Lie(nB''n^{-1}))$, we have $Lie(nB''n^{-1}) = \mathfrak{b}$.

In particular, we have $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_{-\alpha_0} \neq 0$. Since $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{b})$, we have $H^0(w, \mathfrak{g}/\mathfrak{b})_{-\alpha_0} \neq 0$. Hence, we have $w^{-1}(\alpha_0) < 0$.

Proof of the converse follows from Lemma 6.2. □

7. Automorphism group of $Z(w,\underline{i})$:

In this section, we study the automorphism group of a BSDH variety.

Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, let $\underline{i} = (i_1, i_2, \ldots, i_r)$.

Recall that for any reduced expression $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \ldots, j_N)$ and $(j_1, j_2, \ldots, j_r) = \underline{i}$, there exits a natural homomorphism

$$f_w : H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$$

of Lie algebras from Section 5.

Recall the following notation:

$$J'(w,\underline{i}) := \{l \in \{1,2,\ldots,r\} : \langle \alpha_{i_l}, \alpha_{i_k}\rangle = 0 \text{ for all } k < l\}$$

$$J(w,\underline{i}) := \{\alpha_{i_l} : l \in J'(w,\underline{i})\} \subset S.$$

Note that the simple reflections $\{s_{i_j} : j \in J'(w,\underline{i})\}$ commute with each other. For each α in $J(w,\underline{i})$, fix a representative n_α of s_α in $N_G(T)$ and let $P_{J(w,\underline{i})}$ be the subgroup of G generated by B and $\{n_\alpha : \alpha \in J(w,\underline{i})\}$. Let $\mathfrak{p}_{J(w,\underline{i})}$ be the Lie algebra of $P_{J(w,\underline{i})}$.

Then, we have

Theorem 7.1.

(1) $\mathfrak{p}_{J(w_0,\underline{i})} \simeq H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$.
(2) $\mathfrak{p}_{J(w,\underline{i})}$ *is isomorphic to a Lie subalgebra of* $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ *if and only if* $w^{-1}(\alpha_0) < 0$. *In such a case, we have* $\mathfrak{p}_{J(w,\underline{i})} = \mathfrak{p}_{J(w_0,\underline{j})}$ *for any reduced expression* $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ *of* w_0 *such that* $\underline{j} = (j_1, j_2, \ldots, j_N)$ *and* $(j_1, j_2, \ldots, j_r) = \underline{i}$.
(3) *If* G *is simply laced,* $\mathfrak{p}_{J(w,\underline{i})} \simeq H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ *if and only if* $w^{-1}(\alpha_0) < 0$. *In such a case, we have* $\mathfrak{p}_{J(w_0,\underline{j})} \simeq H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$, *where* $\underline{j}$ *is as in (2).*
(4) *If* G *is simply laced,* $f_w : H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ *is surjective, where* $\underline{j}$ *is as in (2).*

Proof. Proof of (1): By Lemma 5.1(2), $f_{w_0} : \mathfrak{b} \longrightarrow H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ is injective. Also, by Corollary 5.2(1), $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ is Lie subalgebra of $\mathfrak{g}$.

By Proposition 6.3, any $\mu \in M_{\geq 0} \setminus \{0\}$ such that $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})_\mu \neq 0$ is of the form $\mu = \alpha_{i_j}$ for some $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k}\rangle = 0$ for all $1 \leq k \leq j-1$. Hence, we conclude that $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ is isomorphic to $\mathfrak{p}_{J(w_0,\underline{i})}$.

Proof of (2): If $\mathfrak{p}_{J(w,\underline{i})}$ is isomorphic to a Lie subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$, then by Proposition 6.4, we have $w^{-1}(\alpha_0) < 0$.

Conversely, assume that $w^{-1}(\alpha_0) < 0$. Let $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ be a reduced expression of w_0 such that $\underline{i} = (j_1, j_2, \ldots, j_r)$. Set $\underline{j} = (j_1, j_2, \ldots, j_N)$. Clearly, $J(w,\underline{i}) \subset J(w_0,\underline{j})$. Hence, we have $\mathfrak{p}_{J(w,\underline{i})} \subset \mathfrak{p}_{J(w_0,\underline{j})}$.

Therefore, by using (1), $\mathfrak{p}_{J(w,\underline{i})}$ is a Lie subalgebra of $H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})})$.

Now, recall the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
\mathfrak{p}_{J(w,\underline{i})} \arrow[r, hook] & H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})}) \arrow[r, hook] \arrow[d, "f_w"] & \mathfrak{g} \\
\mathfrak{b} \arrow[u, hook] \arrow[r, "f_w|_{\mathfrak{b}}"] & H^0(Z(w,\underline{i}), T_{(w,\underline{i})}) &
\end{tikzcd}$$

(see Section 5).

Since the unique B-stable line $\mathfrak{g}_{-\alpha_0}$ in $H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})})$ lies in $\mathfrak{b}$, by commutativity of the above diagram, we conclude that $f_w : H^0(Z(w_0,\underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ is injective if and only if its restriction $f_w|_{\mathfrak{b}}$ to $\mathfrak{b}$ is injective.

Since $w^{-1}(\alpha_0) < 0$, by Lemma 6.2, $f_w|_{\mathfrak{b}}$ to $\mathfrak{b}$ is injective. Hence, by the above arguments,

$$f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

is injective. Therefore, $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_\alpha \neq 0$ for every $\alpha \in J(w_0, \underline{j})$. Thus, we conclude that $J(w_0, \underline{j}) = J(w, \underline{i})$.

Proof of (3): If G is simply laced, by Theorem 2.7 (3), we have $H^0(w, \mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$ if and only if $w^{-1}(\alpha_0) < 0$. Recall from Section 5 that $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{b})$. Hence, from the proof of (2), we conclude that $\mathfrak{p}_{J(w,\underline{i})} \simeq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ if and only if $w^{-1}(\alpha_0) < 0$.

Proof of (4): Proof is by descending induction on $l(w)$. If $w = w_0$, we are done. Otherwise, let $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ be a reduced expression for w_0 such that $(j_1, j_2, \ldots, j_r) = \underline{i}$ and $r \leq N-1$. Let $v = s_{j_1}s_{j_2}\cdots s_{j_{r+1}}$ and let $\underline{i}' = (j_1, j_2, \ldots, j_{r+1})$. Note that $l(w) = l(v) - 1$.

Since G is simply laced, by using LES and Lemma 2.6 (2) we have the following short exact sequence of B-modules:

$$0 \longrightarrow H^0(v, \alpha_{i_{r+1}}) \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(v, \alpha_{i_{r+1}}) = 0.$$

Consider the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \arrow[d, "f_v"] \arrow[dr, "f_w"] & \\
H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}) \arrow[r] & H^0(Z(w, \underline{i}), T_{(w,\underline{i})})
\end{tikzcd}$$

By descending induction on $l(w)$, $f_v : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})$ is surjective. By commutativity of the above diagram and by the above short exact sequence, we conclude that $f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is surjective. This completes the proof of (4). $\square$

Recall that $\leq$ is the Bruhat-Chevalley ordering on W and $supp(w) := \{j \in \{1, 2, \ldots, n\} : s_j \leq w\}$, the support of w. For simplicity of notation we denote $supp(w)$ by A_w. For $j \in A_w$, let n_j be a representative of s_j in $N_G(T)$. Let P_{A_w} be the standard parabolic subgroup of G containing B and $\{n_j : j \in A_w\}$. Let $\mathfrak{p}_{A_w}$ be the Lie algebra of P_{A_w}.

Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression of w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ be a reduced expression for w_0 such that $(j_1, j_2, \ldots, j_r) = \underline{i}$.

Set $J_1 := (\{1, 2, \ldots, n\} \setminus A_w) \cap J'(w_0, \underline{j})$. Let $R_w = R^+ \setminus (\bigcup_{v \leq w} R^+(v^{-1}))$.

Let $f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ be the homomorphism as above.

Now, we will describe the kernel of the map f_w when G is simply laced. Let $Ker(f_w)$ be the kernel of f_w.

Corollary 7.2. *Let G be simply laced. Then, we have*

$$Ker(f_w) = (\bigcap_{k \in A_w} Ker(\alpha_k)) \oplus (\bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}) \oplus (\bigoplus_{j \in J_1} \mathfrak{g}_{\alpha_j}).$$

Proof. Step 1: We will prove that for every $j \in A_w$, the restriction of f_w to the subspace $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is injective.

Fix $j \in A_w$. Let k be the least positive integer in $\{1, 2, \ldots, r\}$ such that $j = i_k$. Let $v = s_{i_1} s_{i_2} \cdots s_{i_k}$ and set $\underline{i}' = (i_1, \ldots, i_k)$. Then, by Lemma 4.3(2), we see that $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is a B_{α_j}-submodule of $H^0(v, \alpha_j)$. By *LES*, $H^0(v, \alpha_j)$ is a B-submodule of $H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})$. Let $g : H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})$ be the homomorphism of B-modules induced by the fibration $Z(w, \underline{i}) \longrightarrow Z(v, \underline{i}')$.

Now, consider the following commutative diagram of B-modules:

$$\begin{tikzcd} H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \arrow[d, "f_w"] \arrow[dr, "f_v"] & \\ H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \arrow[r, "g"] & H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}) \end{tikzcd}$$

Note that $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is a subspace of $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$. Therefore, by the above arguments, the restriction of f_v to the subspace $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is injective. Hence, by commutativity of the above diagram, we conclude that the restriction of f_w to the subspace $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is injective.

Step 2: Let $\mathfrak{l}_{A_w}$ be the Levi subalgebra of $\mathfrak{p}_{A_w}$, let $\mathfrak{z}(\mathfrak{l}_{A_w})$ be the center of $\mathfrak{l}_{A_w}$. We will prove that

$$\mathfrak{h} \cap Ker(f_w) = \mathfrak{z}(\mathfrak{l}_{A_w}) = \bigcap_{k \in A_w} Ker(\alpha_k).$$

First note that $\bigcap_{k \in A_w} Ker(\alpha_k) = \mathfrak{z}(\mathfrak{l}_{A_w})$ and the dimension of $\mathfrak{z}(\mathfrak{l}_{A_w})$ is $n - d(w)$ (since $|A_w| = d(w)$).

Now, we prove that $\mathfrak{h} \cap Ker(f_w)$ is contained in $\bigcap_{k \in A_w} Ker(\alpha_k)$.

Assume the contrary. Then, there exists a $k \in A_w$ and $h \in \mathfrak{h} \cap Ker(f_w)$ such that $\alpha_k(h) \neq 0$. Then,

$$x_{-\alpha_k} \cdot h = -[h, x_{-\alpha_k}] = \alpha_k(h) x_{-\alpha_k}$$

is a non zero multiple of $x_{-\alpha_k}$. Hence $\mathfrak{g}_{-\alpha_k}$ is contained in $Ker(f_w)$, which contradicts step 1. Therefore, $\mathfrak{h} \cap Ker(f_w)$ is contained in $\bigcap_{k \in A_w} Ker(\alpha_k)$.

By Proposition 6.1, we have $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})_0 = \mathfrak{h}$ and $dim(\mathfrak{h} \cap Ker(f_w)) = n - d(w)$. Hence, we see that

$$f_w(H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})_0) = H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0.$$

By the above arguments, $\mathfrak{h} \cap Ker(f_w)$ is a subspace of $\bigcap_{k \in A_w} Ker(\alpha_k)$ having the same dimension as that of $\bigcap_{k \in A_w} Ker(\alpha_k)$. Hence, we conclude that

$$\mathfrak{h} \cap Ker(f_w) = \bigcap_{k \in A_w} Ker(\alpha_k) = \mathfrak{z}(\mathfrak{l}_{A_w}).$$

Step 3: We will prove that for $j \in J_1$, sl_{2,α_j} is contained in $Ker(f_w)$.

Fix $j \in J_1$. By Theorem 7.1(2), it follows that sl_{2,α_j} is a B_{α_j}-submodule of $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$. By Proposition 6.3(1), we see that $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{\alpha_j} = 0$. Hence, $\mathfrak{g}_{\alpha_j} \subset Ker(f_w)$. Since sl_{2,α_j} is a cyclic B_{α_j}-module generated by $\mathfrak{g}_{\alpha_j}$, it follows that sl_{2,α_j} is contained in $Ker(f_w)$.

Step 4: The intersection of the nilradical of $\mathfrak{b}$ and $Ker(f_w)$ is equal to the direct sum $\bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}$ of T-modules.

Consider the birational morphism $\phi_w : Z(w, \underline{i}) \longrightarrow X(w)$. Note that ϕ_w is a B-equivariant morphism for the natural left action of B on $Z(w, \underline{i})$ (respectively, on $X(w)$). Let $\phi : B \longrightarrow Aut^0(X(w))$ (respectively, $\phi' : B \longrightarrow Aut^0(Z(w, \underline{i}))$) be the homomorphism induced by the action of B on $X(w)$ (respectively, on $Z(w, \underline{i})$). Since ϕ_w is birational, we have $Ker(\phi) \cap B_u = Ker(\phi') \cap B_u$, where B_u is the unipotent radical of B.

Since G is simply laced, by [16, Corollary 3.9], we conclude that $\mathfrak{b}_u \cap Ker(f_w) = \bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}$, where $\mathfrak{b}_u$ is the nilradical of $\mathfrak{b}$.

From the steps 1 to 4, we conclude that

$$Ker(f_w) = (\bigcap_{k \in A_w} Ker(\alpha_k)) \oplus (\bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}) \oplus (\bigoplus_{j \in J_1} \mathfrak{g}_{\alpha_j}).$$

□

Recall that if X is a smooth projective variety over $\mathbb{C}$, the connected component of the group of all automorphisms of X containing identity automorphism is an algebraic group (see [17, p.17, Theorem 3.7], [8, p.268], which also deals the case when X may be singular or it may be defined over any field). Futher, the Lie algebrs of this automorphism group is isomorphic to the space of all vector fields on X, that is the space $H^0(X, T_X)$ of all global sections of the tangent bundle T_X of X (see [17, p.13, Lemma 3.4]).

We now prove the main results of the paper using Theorem 7.1.

Recall that $Aut^0(Z(w, \underline{i}))$ is the connected component of the identity element of the automorphism group of $Z(w, \underline{i})$.

Theorem 7.3.

(1) *$P_{J(w_0,\underline{i})} \simeq Aut^0(Z(w_0, \underline{i}))$.*
(2) *$Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, we have $P_{J(w,\underline{i})} = P_{J(w_0,\underline{j})}$ for any reduced expression $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \cdots, j_N)$ and $(j_1, j_2, \cdots, j_r) = \underline{i}$.*
(3) *If G is simply laced, $P_{J(w,\underline{i})} \simeq Aut^0(Z(w, \underline{i}))$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, we have $Aut^0(Z(w, \underline{i})) \simeq Aut^0(Z(w_0, \underline{j}))$, where $\underline{j}$ is as in (2).*
(4) *The homomorphism $f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is induced by a homomorphism $g_w : Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups, where $\underline{j}$ is as in (2).*
(5) *If G is simply laced, the homomorphism $g_w : Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups is surjective, where $\underline{j}$ is as in (2).*
(6) *The rank of $Aut^0(Z(w, \underline{i}))$ is at most the rank of G.*

Proof. Recall that by [17, Theorem 3.7], $Aut^0(Z(w,\underline{i}))$ is an algebraic group and

$$Lie(Aut^0(Z(w,\underline{i}))) = H^0(Z(w,\underline{i}), T_{(w,\underline{i})}).$$

Let $\pi : \widetilde{G} \longrightarrow G$ be the simply connected covering of G. Let $\widetilde{P}_{J(w,\underline{i})}$ (respectively, $\widetilde{B}$) be the inverse image of $P_{J(w,\underline{i})}$ (respectively, of B) in $\widetilde{G}$.

Proof of (2): If $w^{-1}(\alpha_0) < 0$, then by Theorem 7.1(2), $\mathfrak{p}_{J(w,\underline{i})}$ is isomorphic to a Lie subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$. Hence, there is a homomorphism $\widetilde{\psi}_w : \widetilde{P}_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups. Since the center $Z(\widetilde{P}_{J(w,\underline{i})})$ of $\widetilde{P}_{J(w,\underline{i})}$ is same as $Z(\widetilde{B})$ and B acts on $Z(w,\underline{i})$, $Z(\widetilde{P}_{J(w,\underline{i})})$ acts trivially on $Z(w,\underline{i})$. Hence, the action of $\widetilde{P}_{J(w,\underline{i})}$ induces a homomorphism $\psi_w : P_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups. Since $\mathfrak{p}_{J(w,\underline{i})}$ is isomorphic to a Lie subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$, ψ_w is an isomorphism onto its image.

On the other hand, if $Aut^0(Z(w,\underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$, then there is an injective homomorphism $\psi_w : P_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups. Further, ψ_w induces an injective homomorphism $\widetilde{f}_w : \mathfrak{p}_{J(w,\underline{i})} \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ of Lie algebras. Hence, by Theorem 7.1(2), we have $w^{-1}(\alpha_0) < 0$. This completes the proof of (2).

Proofs of (1), (3) and (4) are similar to the proof of (2). For the sake of completeness we give proof here.

Proof of (1). By Theorem 7.1(1), $\mathfrak{p}_{J(w_0,\underline{i})}$ is isomorphic to the Lie algebra $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$. Hence, there is a homomorphism $\widetilde{\psi}_{w_0} : \widetilde{P}_{J(w_0,\underline{i})} \longrightarrow Aut^0(Z(w_0,\underline{i}))$ of algebraic groups. Since the center $Z(\widetilde{P}_{J(w_0,\underline{i})})$ of $\widetilde{P}_{J(w_0,\underline{i})}$ is same as $Z(\widetilde{B})$ and B acts on $Z(w_0,\underline{i})$, $Z(\widetilde{P}_{J(w_0,\underline{i})})$ acts trivially on $Z(w_0,\underline{i})$. Hence, the action of $\widetilde{P}_{J(w_0,\underline{i})}$ induces a homomorphism $\psi_{w_0} : P_{J(w_0,\underline{i})} \longrightarrow Aut^0(Z(w_0,\underline{i}))$ of algebraic groups. Note that ψ_{w_0} induces an isomorphism $\widetilde{f}_{w_0} : \mathfrak{p}_{J(w_0,\underline{i})} \longrightarrow H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ of Lie algebras. Hence, we conclude that $\psi_{w_0} : P_{J(w_0,\underline{i})} \longrightarrow Aut^0(Z(w_0,\underline{i}))$ is an isomorphism of algebraic groups.

Proof of (3). By (2), we have the homomorphism $\psi_w : P_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups is injective if and only if $w^{-1}(\alpha_0) < 0$. Since G is simply laced, by Theorem 7.1(3), we conclude the proof of (3).

Proof of (4). By (1), we have $P_{J(w_0,\underline{j})} \simeq Aut^0(Z(w_0,\underline{j}))$.

Let

$$P_{J(w_0,\underline{j})} = LP_u = L_{ss}Z(L)P_u$$

be the Levi decomposition of $P_{J(w_0,\underline{j})}$ such that $T \subset L$, where L is the Levi factor of $P_{J(w_0,\underline{j})}$ containing T, L_{ss} is semi simple part of L and P_u is unipotent radical of $P_{J(w_0,\underline{j})}$.

Since $P_u \subset B$, we have the homomorphism $f_1 : P_u \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups.

Since $Z(L) \subset T \subset B$, we have the homomorphism $f_2 : Z(L) \longrightarrow Aut^0(Z(w,\underline{i}))$.

For $j \in J(w,\underline{i})$, by Lemma 6.3, sl_{2,α_j} is contained in $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$. Hence for each $j \in J(w,\underline{i})$, we have $\phi_j : SL_{2,\alpha_j} \longrightarrow Aut^0(Z(w,\underline{i}))$.

For $j \in J(w_0, \underline{j}) \setminus J(w, \underline{i})$, by the proof of Corollary 7.2 (even though G is not necessarily simply laced), we have $\mathfrak{g}_{\alpha_j} \subset Ker(f_w)$. Hence, the homomorphism $\phi_j : SL_{2,\alpha_j} \longrightarrow Aut^0(Z(w, \underline{i}))$ is trivial. That is SL_{2,α_j} acts trivially on $Z(w, \underline{i}))$ for each $j \in J(w_0, \underline{j}) \setminus J(w, \underline{i})$.

Therefore, we have the homomorphism $\widetilde{L} \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups, where $\widetilde{L}$ is inverse image of L in $\widetilde{G}$ by the universal cover $\pi : \widetilde{G} \longrightarrow G$.

Claim: For $j \in J(w_0, \underline{j})$, we have the following commutative diagram of algebraic groups:

$$\begin{tikzcd} SL_{2,\alpha_j} \arrow[r, "\phi_j"] \arrow[d] & Aut^0(Z(w, \underline{i})) \\ PGL_{2,\alpha_j} \arrow[ur] & \end{tikzcd}$$

Let G_{α_j} be the image of SL_{2,α_j} in $Aut^0(Z(w, \underline{i}))$, let $B_{\alpha_j} = B \cap G_{\alpha_j}$. Let $\widetilde{B}_{\alpha_j} = \pi^{-1}(B_{\alpha_j})$, which is a Borel subgroup of SL_{2,α_j}.

Now consider the following commutative diagram:

$$\begin{tikzcd} SL_{2,\alpha_j} \arrow[r, "\phi_j"] & Aut^0(Z(w, \underline{i})) \\ \widetilde{B}_{\alpha_j} \arrow[r, "\pi"] \arrow[u, hook] & B_{\alpha_j} \arrow[u, hook] \end{tikzcd}$$

Since the kernel of π is contained in the kernel of ϕ_j, the action of $Z(\widetilde{B}_{\alpha_j})$ on $Z(w, \underline{i})$ is trivial. Since $Z(\widetilde{B}_{\alpha_j}) = Z(SL_{2,\alpha_j})$, we have the homomorphism $PSL_{2,\alpha_j} \longrightarrow Aut^0(Z(w, \underline{i}))$. This proves the claim.

From the above discussion, we conclude that the center $Z(\widetilde{P}_{J(w_0,\underline{j})})$ acts trivially on $Z(w, \underline{i})$. Hence, there is a homomorphism $g_w : P_{J(w_0,\underline{j})} \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups which induces f_w. This completes the proof of (4).

Proof of (5) follows from Theorem 7.1(4).

Proof of (6) follows from Proposition 6.1. $\square$

We use the same notation as before. Assume that G is simply laced.

Let $g_w : Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i}))$ be the natural map as in Theorem 7.3 (4). Let U^+ be the unipotent radical of B^+. For $j \in J_1$, let $U^+_{\alpha_j}$ denote the one-dimensional T-stable closed subgroup of U^+ (for the conjugation action of T on G) corresponding to α_j. Let $T(w) := \bigcap_{k \in A_w} Ker(\alpha_k)$. Since $\{\alpha_k : k \in A_w\}$ is a subset of the $\mathbb{Z}$-basis S of $X(T)$, $T(w)$ is connected.

Corollary 7.4. *The connected component of the kernel of the map g_w is the closed subgroup of $Aut^0(Z(w_0, \underline{j}))$ generated by the torus $T(w)$, $\{U_{-\beta} : \beta \in R_w\}$ and $\{U^+_{\alpha_j} : j \in J_1\}$.*

Proof. Let K be the kernel of the homomorphism g_w. Then, we have the following exact sequence of algebraic groups:

$$1 \longrightarrow K \longrightarrow Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i})) \longrightarrow 1.$$

By using the differentials, we have following exact sequence of Lie algebras:

$$0 \longrightarrow Lie(K) \longrightarrow H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow 0.$$

By [12, p.85, Theorem 12.5], the Lie algebra of K is $Ker(f_w)$. By Corollary 7.2, we have

$$Ker(f_w) = (\bigcap_{k \in A_w} Ker(\alpha_k)) \oplus (\bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}) \oplus (\bigoplus_{j \in J_1} \mathfrak{g}_{\alpha_j}).$$

Let H be the closed subgroup of $Aut^0(Z(w_0, \underline{j}))$ generated by $T(w)$, $\{U_{-\beta} : \beta \in R_w\}$ and $\{U^+_{\alpha_j} : j \in J_1\}$. Note that H is connected (see [12, p.56, Corollary 7.5]) and $Lie(H) \subset Ker(f_w)$. Since $dim(Lie(H)) = dim(Ker(f_w))$, we have

$$Lie(H) = Ker(f_w).$$

Hence, we conclude that $K^0 = H$. This completes the proof of the corollary. $\square$

In the following corollary, for the simplicity of notation we denote the homogeneous vector bundle $\mathcal{L}(w, \mathbb{C}_{\alpha_0})$ on $X(w)$ corresponding to the character α_0 of B by $\mathcal{L}_{\alpha_0}$.

Consider the left action of T on G/B. Let $w \in W$. Note that the Schubert variety $X(w^{-1})$ is T-stable. We use the notion of semi-stable points introduced by Mumford [18]. Let α_0 be the highest root of G with respect to T and B^+. We denote by $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ the set of all semi-stable points of $X(w^{-1})$ with respect to the T-linearized line bundle $\mathcal{L}_{\alpha_0}$ corresponding to the character α_0 of B (see [18]).

The following result is a formulation of the Theorem 7.3 using semi-stable points.

Corollary 7.5. (1) *$Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$ if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$.*
(2) *If G is simply laced, $Aut^0(Z(w, \underline{i})) \simeq P_{J(w,\underline{i})}$ if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$.*

Proof. By [15, Lemma 2.1], we have $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$ if and only if $w^{-1}(\alpha_0) < 0$. Proof of the corollary follows from Theorem 7.3 (2) and Theorem 7.3 (3). $\square$

Remark: By Theorem 7.3, the automorphism group of the BSDH-variety $Z(w, \underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w.

Example: Let $G = PSL(4, \mathbb{C})$. Consider the following different reduced expressions for w_0:

(1) $(w_0, \underline{i}_1) = s_1s_2s_1s_3s_2s_1$, $J(w_0, \underline{i}_1) = \{\alpha_1\}$.
(2) $(w_0, \underline{i}_2) = s_2s_1s_2s_3s_2s_1$, $J(w_0, \underline{i}_2) = \{\alpha_2\}$.
(3) $(w_0, \underline{i}_3) = s_3s_2s_3s_1s_2s_3$, $J(w_0, \underline{i}_3) = \{\alpha_3\}$.
(4) $(w_0, \underline{i}_4) = s_1s_3s_2s_3s_1s_2$, $J(w_0, \underline{i}_4) = \{\alpha_1, \alpha_3\}$.

By Theorem 7.3, we see that $Aut^0(Z(w_0, \underline{i}_1))$, $Aut^0(Z(w_0, \underline{i}_2))$, $Aut^0(Z(w_0, \underline{i}_3))$ and $Aut^0(Z(w_0, \underline{i}_4))$ are isomorphic to $P_{\{\alpha_1\}}, P_{\{\alpha_2\}}, P_{\{\alpha_3\}}, P_{\{\alpha_1,\alpha_3\}}$ respectively.

Therefore, we observe that $Aut^0(Z(w_0, \underline{i}_1))$ and $Aut^0(Z(w_0, \underline{i}_4))$ are not isomorphic and hence we conclude that the BSDH-varieties $Z(w_0, \underline{i}_1)$ and $Z(w_0, \underline{i}_4)$ are not isomorphic. Also, we observe that $Z(w_0, \underline{i}_1)$ and $Z(w_0, \underline{i}_2)$ are not isomorphic as $P_{\{\alpha_1\}}$ and $P_{\{\alpha_2\}}$ are not isomorphic.

Remark: Even if the automorphism groups of the BSDH-varieties are isomorphic, it is not clear that the BSDH-varieties are isomorphic.

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